Operations Research, Spring 2015
Homework 2

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April 21, 2015

1 Problems

1. (10 points) A company ships 25000 products per month to a customer. The products may be produced at three different plants. The production capacity, fixed monthly cost of operation, and variable cost per unit at each plant are given in the table below. The fixed cost for a plant is incurred if and only if the plant is used to make any product. Formulate an IP whose solution minimizes the monthly costs of meeting the customer' demand.

Plant	Fixed cost ($)	Variable cost ($)	Capacity
1	70000	20	20000
2	40000	30	18000
3	50000	35	25000

2. (15 points) Continue from Problem 1. The company now gets another customer. The original customer, customer 1, still orders 25000 units of that product. However, the new customer, customer 2, orders another product. Let's call the product ordered by customer i as product i, $i = 1, 2$. Each month customer 2 orders 20000 units of product 2. The unit processing times of producing products 1 and 2 are the same at each plant.[1] However, product 2 requires different fixed and variable production costs at each plant:

Plant	Fixed cost ($)	Variable cost ($)
1	20000	16
2	30000	14
3	15000	18

   (a) (10 points) Formulate an IP whose solution minimizes the monthly costs of meeting both customers' demands.

   (b) (5 points) Use AMPL to solve the company's problem. Make concrete managerial suggestions. DO NOT submit your AMPL program; just submit your suggestions.

3. (20 points; 5 points each) Consider the following knapsack problem

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 5x_3 + 4x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 4x_3 + 4x_4 \leq 8 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 4. \end{aligned}$$

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[1] For example, one may produce 15000 units of product 1 and 5000 units of product 2 in plant 1. Alternatively, she may produce 18000 units of product 1 and 2000 units of product 2 in plant 1.

(a) Find its linear relaxation. Then solve the linear relaxation by the value-weight ratio sorting algorithm discussed in class.
(b) Find the dual LP of the linear relaxation in Part (a). Apply complementary slackness to determine which dual constraints must be binding.
(c) Using those binding constraints to remove some dual variables (one binding constraint allows you to remove one variable). Then graphically solve the remaining dual LP.
(d) Do your solutions obtained in Part (a) and (c) satisfy dual theorems? If yes, is it a coincidence? If no, why? Explain.

4. (10 points) Use the branch-and-bound algorithm to solve the knapsack problem in Problem 3.

5. (45 points) The classic *vertex cover* problem is defined as follows. Given an undirected graph $G = (V, E)$, where V is the set of nodes and E is the set of edges, and node weights $w_i > 0$ for node $i \in V$, we want to select a subset of nodes such that all edges are adjacent to at least one selected node and the total weight of selected nodes is minimized. As an example, for $G_1 = (V_1, E_1) = (\{1, 2, 3\}, \{[1, 2], [2, 3], [1, 3]\}$ with weights $w_1 = 2$, $w_2 = 3$, and $w_3 = 10$, selecting any two nodes forms a vertex cover and the optimal vertex cover is $\{1, 2\}$. Selecting only one node, however, is not a vertex cover as one edge will be left uncovered.

(a) (5 points) For the graph G_1, formulate an IP whose optimal solution is a minimum-weight vertex cover.
(b) (5 points) Find the linear relaxation of the IP in Part (a). Prove that when you relax the binary constraints, those "no-greater-than-1" constraints are redundant. Let's denote the LP without those "no-greater-than-1" constraints by (P).
(c) (5 points) Find the standard form for (P). Prove or disprove that the coefficient matrix is totally unimodular.
(d) (10 points) For (P), analytically determine whether the solution $x^* = (x_1^*, x_2^*, x_3^*) = (1, 1, 0)$ is optimal. Please note that (P) is an LP; showing that x^* is better than $(1, 0, 1)$ and $(0, 1, 1)$ is not enough!

Note. Using AMPL does not *analytically* prove anything. However, it does give you a direction to write your proof.

Hint. Solving (P) with the simplex method does provide an analytical proof. However, you do not need to do that; finding the reduced costs for x^* is good enough.
(e) (10 points) Do your answers in Parts (c) and (d) conflict with each other? Explain.
(f) (5 points) For G_1, assign different values to w_is so that an LR-optimal solution is not IP-feasible. Justify your solution.
(g) (5 points) Use the branch-and-bound algorithm to solve the IP in (a).

6. (15 bonus points; 5 points each) Consider the vertex cover problem defined on a general undirected graph $G = (V, E)$.

(a) Formulate an IP whose solution is the minimum-weight vertex cover.
(b) For the linear relaxation of the IP in (a), remove those redundant "no-greater-than-1" constraints to obtain a primal LP. Then find the dual LP.

Hint. Given a node $i \in V$, $\{[i, j] : [i, j] \in E\}$ is the set of edges adjacent to node i.
(c) Is the primal or dual LP easier to be solved by the simplex method? Explain.

2 Submission rules

The deadline of this homework is 2pm, April 27, 2015. Please put a hard copy of the work into the instructor's mailbox on the first floor of the Management Building 2 by the due time. Works submitted between 2pm and 3pm will get 10 points deducted as a penalty. Submissions later than 3pm will not be accepted. Each student must submit her/his individual work.