Operations Research

The Simplex Method (Part 1)

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Introduction

- In these two lectures, we will study how to **solve** an LP.
- The algorithm we will introduce is **the simplex method**.
  - Developed by **George Dantzig** in 1947.
  - Opened the whole field of Operations Research.
  - Implemented in most commercial LP solvers.
  - **Very efficient** for almost all practical LPs.
  - With **very simple ideas**.
- The method is general in an indirect manner.
  - There are many different forms of LPs.
  - We will first show that each LP is equivalent to a **standard form** LP.
  - Then we will show how to solve standard form LPs.
- Read Sections 4.1 to 4.4 of the textbook thoroughly!
- These two lectures will be full of **algebra** and **theorems**. Get ready!

Road map

- **Standard form LPs**.
- Basic solutions.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.

Standard form LPs

- First, let's define the **standard form**.[1]

Definition 1 (Standard form LP)

An LP is in the standard form if

- *all the RHS values are nonnegative,*
- *all the variables are nonnegative, and*
- *all the constraints are equalities.*

- RHS = right hand sides. For any constraint

$$g(x) \leq b, \quad g(x) \geq b, \quad \text{or } g(x) = b,$$

  b is the RHS.
- There is no restriction on the objective function.

[1] In the textbook, this form is called the augmented form. In the world of OR, however, "standard form" is a more common name for LPs in this format.

Finding the standard form

- How to find the standard form for an LP?
- Requirement 1: **Nonnegative RHS**.
  - If it is negative, **switch** the LHS and the RHS.
  - E.g.,

$$2x_1 + 3x_2 \leq -4$$

is equivalent to

$$-2x_1 - 3x_2 \geq 4.$$

Finding the standard form

- Requirement 2: **Nonnegative variables**.
  - If x_i is **nonpositivie**, replace it by $-x_i$. E.g.,

$$2x_1 + 3x_2 \leq 4, x_1 \leq 0 \quad \Leftrightarrow \quad -2x_1 + 3x_2 \leq 4, x_1 \geq 0.$$

  - If x_i is **free**, replace it by $x_i' - x_i''$, where $x_i', x_i'' \geq 0$. E.g.,

$$2x_1 + 3x_2 \leq 4, x_1 \text{ urs.} \quad \Leftrightarrow \quad 2x_1' - 2x_1'' + 3x_2 \leq 4, x_1' \geq 0, x_1'' \geq 0.$$

$x_i = x_i' - x_i''$	$x_i' \geq 0$	$x_i'' \geq 0$
5	5	0
0	0	0
-8	0	8

Finding the standard form

- Requirement 3: **Equality constraints**.
  - For a "$\leq$" constraint, **add a slack** variable. E.g.,

$$2x_1 + 3x_2 \leq 4 \quad \Leftrightarrow \quad 2x_1 + 3x_2 + x_3 = 4, \quad x_3 \geq 0.$$

  - For a "$\geq$" constraint, **minus a surplus/excess** variable. E.g.,

$$2x_1 + 3x_2 \geq 4 \quad \Leftrightarrow \quad 2x_1 + 3x_2 - x_3 = 4, \quad x_3 \geq 0.$$

  - For ease of exposition, they will both be called slack variables.
  - A slack variable measures the **gap** between the LHS and RHS.

An example

$$
\begin{array}{rrcrcrcr}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & x_1 & - & x_2 & & & \geq & -8 \\
 & 2x_1 & + & x_2 & + & x_3 & = & 9 \\
 & \multicolumn{7}{c}{x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ urs.}}
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{rrcrcrcr}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & -x_1 & + & x_2 & & & \leq & 8 \\
 & 2x_1 & + & x_2 & + & x_3 & = & 9 \\
 & \multicolumn{7}{c}{x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ urs.}}
\end{array}
$$

An example

$$
\begin{array}{rrrrrrrrrrr}
 & \min & 3x_1 & - & 2x_2 & + & 4x_3 & - & 4x_4 & & \\
\rightarrow & \text{s.t.} & x_1 & - & 2x_2 & - & x_3 & + & x_4 & \geq & 6 \\
 & & -x_1 & - & x_2 & & & & & \leq & 8 \\
 & & 2x_1 & - & x_2 & + & x_3 & - & x_4 & = & 9 \\
 & & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4}
\end{array}
$$

$$
\begin{array}{rrrrrrrrrrrrrrr}
 & \min & 3x_1 & - & 2x_2 & + & 4x_3 & - & 4x_4 & & & & & & \\
\rightarrow & \text{s.t.} & x_1 & - & 2x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 6 \\
 & & -x_1 & - & x_2 & & & & & & & + & x_6 & = & 8 \\
 & & 2x_1 & - & x_2 & + & x_3 & - & x_4 & & & & & = & 9 \\
 & & \multicolumn{13}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.}
\end{array}
$$

Standard form LPs in matrices

- Given **any** LP, we may find its standard form.
- With matrices, a standard form LP is expressed as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- E.g., for

$$\begin{array}{llllllllll} \min & 2x_1 & - & x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 5x_2 & + & x_3 & & & = & 5 \\ & 3x_1 & - & 6x_2 & & & + & x_4 & = & 4 \\ & & & x_i \geq 0 & \forall i = 1, ..., 4, & & & & & \end{array}$$

$$c = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{bmatrix}.$$

- We will denote the number of constraints and variables as m and n.
  - $A \in \mathbb{R}^{m \times n}$ is called the **coefficient matrix**.
  - $b \in \mathbb{R}^m$ is called the **RHS vector**.
  - $c \in \mathbb{R}^n$ is called the **objective vector**.
- The objective function can be either max or min.

Solving standard form LPs

- So now we only need to find a way to solve standard form LPs.
- How?
- A standard form LP is still an LP.
- If it has an optimal solution, it has an **extreme point** optimal solution! Therefore, we only need to search among extreme points.
- Our next step is to understand more about the extreme points of a standard form LP.

Road map

- Standard form LPs.
- **Basic solutions**.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.

Bases

- Consider a standard form LP with m constraints and n variables

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array}$$

  - We may assume that rank $A = m$, i.e., all rows of A are independent.[2]
  - This then implies that $m \leq n$. As the problem with $m = n$ is trivial, we will assume that $m < n$.
- For the system $Ax = b$, now there are more columns than rows. Let's select some columns to form a **basis**:

Definition 2 (Basis)

A basis B of a standard form LP is a selection of m variables such that A_B, the matrix formed by the m corresponding columns of A, is invertible/nonsingular.

[2] This assumption is without loss of generality. Why?

Basic solutions

- By ignoring the other $n - m$ variables, $Ax = b$ will have a unique solution (because A_B is invertible).
- Each basis uniquely defines a **basic solution**:

Definition 3 (Basic solution)

A basic solution to a standard form LP is a solution that (1) has $n - m$ variables being equal to 0 and (2) satisfies $Ax = b$.

  - The $n - m$ variables chosen to be zero are **nonbasic variables**.
  - The remaining m variables are **basic variables**. They form a basis (i.e., A_B^{-1} is invertible; otherwise $Ax = b$ has no solution).
- We use $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$ to denote basic and nonbasic variables, respectively, with respect to a given basis B.
  - We have $x_N = 0$ and $x_B = A_B^{-1}b$.
  - Note that a basic variable may be positive, negative, or zero!

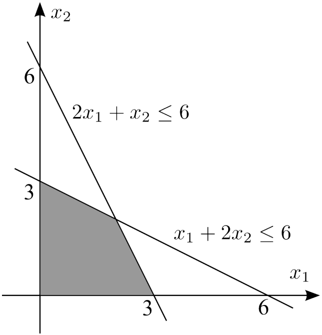
Basic solutions: an example

- Consider an original LP

$$\begin{array}{rrcrcl} \min & 6x_1 & + & 8x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 6 \\ & 2x_1 & + & x_2 & \leq & 6 \\ & \multicolumn{5}{c}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

and its standard form

$$\begin{array}{rrcrcrcrcl} \min & 6x_1 & + & 8x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & 2x_1 & + & x_2 & & & + & x_4 & = & 6 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$



Basic solutions: an example

- In the standard form, $m = 2$ and $n = 4$.
  - There are $n - m = 2$ nonbasic variables.
  - There are $m = 2$ basic variables.
- Steps for obtaining a basic solution:
  - Determine a set of m basic variables to form a basis B.
  - The remaining variables form the set of nonbasic variables N.
  - Set nonbasic variables to zero: $x_N = 0$.
  - Solve the m by m system $A_B x_B = b$ for the values of basic variables.
- For this example, we will solve a two by two system for each basis.

Basic solutions: an example

- The two equalities are

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ 2x_1 & + & x_2 & & & + & x_4 & = & 6. \end{array}$$

- Let's try $B = \{x_1, x_2\}$ and $N = \{x_3, x_4\}$:

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 6 \\ 2x_1 & + & x_2 & = & 6. \end{array}$$

  The solution is $(x_1, x_2) = (2, 2)$. Therefore, the basic solution associated with this basis B is $(x_1, x_2, x_3, x_4) = (2, 2, 0, 0)$.
- Let's try $B = \{x_2, x_3\}$ and $N = \{x_1, x_4\}$:

$$\begin{array}{rcrcl} 2x_2 & + & x_3 & = & 6 \\ x_2 & & & = & 6. \end{array}$$

  As $(x_2, x_3) = (6, -6)$, the basic solution is $(x_1, x_2, x_3, x_4) = (0, 6, -6, 0)$.

Bases

- In general, as we need to choose m out of n variables to be basic, we have **at most** $\binom{n}{m}$ different bases.[3]
- In this example, we have exactly $\binom{4}{2} = 6$ bases.
- By examining all the six bases one by one, we may find all those associated basic variables:

Basis	Basic solution			
	x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	2	2	0	0
$\{x_1, x_3\}$	3	0	3	0
$\{x_1, x_4\}$	6	0	0	-6
$\{x_2, x_3\}$	0	6	-6	0
$\{x_2, x_4\}$	0	3	0	3
$\{x_3, x_4\}$	0	0	6	6

[3] Why "at most"? Why not "exactly"?

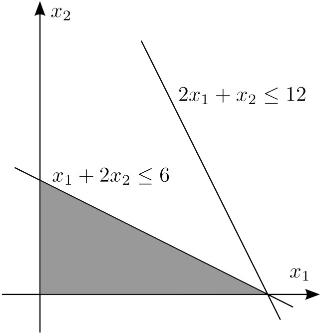
Basic solutions v.s. bases

- For a basis, what matters are **variables**, not **values**.
- Consider another example

$$
\begin{array}{rrcrcr}
\min & 6x_1 & + & 8x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \leq & 6 \\
 & 2x_1 & + & x_2 & \leq & 12 \\
 & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2}
\end{array}
$$

and its standard form

$$
\begin{array}{rrcrcrcrcr}
\min & 6x_1 & + & 8x_2 & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = & 12 \\
 & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.}
\end{array}
$$



Basic solutions v.s. bases

- The six bases and the associated basic variables are listed below:

Basis	Basic solution			
	x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	**6**	**0**	0	0
$\{x_1, x_3\}$	**6**	0	**0**	0
$\{x_1, x_4\}$	**6**	0	0	**0**
$\{x_2, x_3\}$	0	12	-18	0
$\{x_2, x_4\}$	0	3	0	9
$\{x_3, x_4\}$	0	0	6	12

  - Three different bases result in **the same** basic solution!
  - There are six distinct bases but only four distinct basic solutions.
- Number of distinct basic solutions $\leq$ number of distinct bases $\leq \binom{n}{m}$.
- When multiple bases correspond to one single basic solution, the LP is **degenerate**; otherwise, it is **nondegenerate**.
- We will discuss degeneracy only at the end of the next lecture.

Road map

- Standard form LPs.
- Basic solutions.
- **Basic feasible solutions**.
- The geometry of the simplex method.
- The algebra of the simplex method.

Basic feasible solutions

- Among all basic solutions, some are feasible.
  - By the definition of basic solutions, they satisfy $Ax = b$.
  - If one also **satisfies $x \geq 0$**, it satisfies all constraints.
- In this case, it is called **basic feasible solutions** (bfs).[4]

Definition 4 (Basic feasible solution)

A basic feasible solution to a standard form LP is a basic solution whose basic variables are all nonnegative.

- Which are bfs?

Basis	Basic solution			
	x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	2	2	0	0
$\{x_1, x_3\}$	3	0	3	0
$\{x_1, x_4\}$	6	0	0	-6
$\{x_2, x_3\}$	0	6	-6	0
$\{x_2, x_4\}$	0	3	0	3
$\{x_3, x_4\}$	0	0	6	6

[4] In the textbook, the abbreviation is "BF solutions".

Basic feasible solutions and extreme points

- Why bfs are important?
- They are just extreme points!

Proposition 1 (Extreme points and basic feasible solutions)

For a standard form LP, a solution is an extreme point of the feasible region if and only if it is a basic feasible solution to the LP.

Proof. Beyond the scope of this course. □

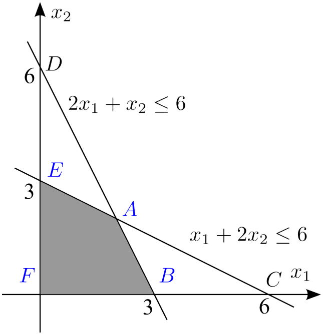
- Though we cannot prove it here, let's get some intuitions with graphs.[5]

[5] Please note that these "intuitions" are never rigorous.

An example

- There is a one-to-one mapping between bfs and extreme points.

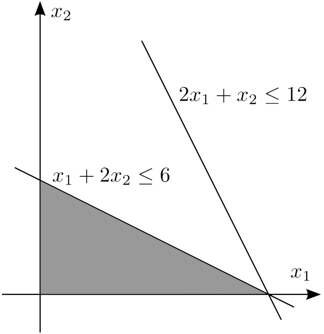
Basis	Bfs?	Point	Basic solution			
			x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	Yes	A	2	2	0	0
$\{x_1, x_3\}$	Yes	B	3	0	3	0
$\{x_1, x_4\}$	No	C	6	0	0	-6
$\{x_2, x_3\}$	No	D	0	6	-6	0
$\{x_2, x_4\}$	Yes	E	0	3	0	3
$\{x_3, x_4\}$	Yes	F	0	0	6	6



Another example

- Would you find the one-to-one correspondence?

Basis	Basic solution			
	x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	6	0	0	0
$\{x_1, x_3\}$	6	0	0	0
$\{x_1, x_4\}$	6	0	0	0
$\{x_2, x_3\}$	0	12	-18	0
$\{x_2, x_4\}$	0	3	0	9
$\{x_3, x_4\}$	0	0	6	12



Optimality of basic feasible solutions

- What's the implication of the previous proposition?

Proposition 2 (Optimality of basic feasible solutions)

For a standard form LP, if there is an optimal solution, there is an optimal basic feasible solution.

Proof. We know if there is an optimal solution, there is an optimal extreme point solution. Moreover, we know extreme points are just bfs. The proof then follows. □

Solving standard form LPs

- To find an optimal solution:
  - Instead of searching among all extreme points, we may search among **all bfs**.
- But the two sets are equally large! What is the difference?
  - Extreme points are defined with **geometry** but bfs are with **algebra**.
  - Checking whether a solution is an extreme point is hard (for a computer).
  - Checking whether a solution is basic feasible is easy (for a computer).
- Given an LP:
  - Enumerating all extreme points is hard.
  - Enumerating all bfs is possible.

Solving standard form LPs

- We are now closer to solve a general LP:
  - We may enumerate all the bfs, compare them, and find the best one.
  - If this LP has an optimal solution, that best bfs is optimal.
- Unfortunately:
  - For a standard form LP with n variables and m constraints, we have at most $\binom{n}{m}$ bfs. Listing them takes too much time![6]
- We need to improve the **search** procedure.
  - We need to analyze bfs more deeply.
  - We need to understand how they are **connected**.
- Let's define **adjacent** bfs.

[6] The complexity is $O(\binom{n}{m}) = O(n!)$; it is an exponential-time algorithm.

Adjacent basic feasible solutions

- Two bfs are either **adjacent** or not:

Definition 5 (Adjacent bases and bfs)

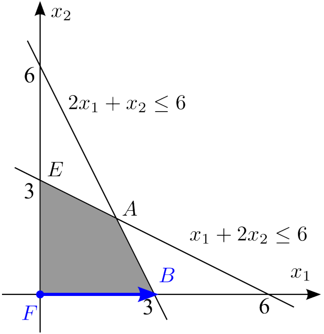
Two bases are adjacent if exactly one of their variable is different. Two bfs are adjacent if their associated bases are adjacent.

  - $\{x_1, x_2\}$ and $\{x_1, x_4\}$ are adjacent.
  - $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are not adjacent.
  - How about $\{x_1, x_2\}$ and $\{x_2, x_4\}$?

Adjacent basic feasible solutions

- A pair of adjacent bfs corresponds to a pair of "adjacent" extreme points, i.e., extreme points that are on **the same edge**.
- Switching from a bfs to its adjacent bfs is **moving along an edge**.

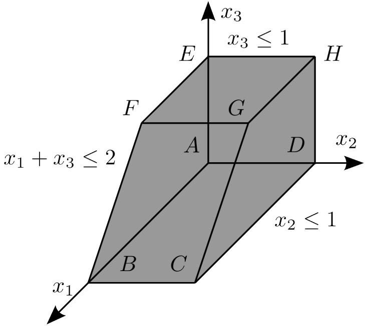
Basis	Point	Basic solution			
		x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	A	2	2	0	0
$\{x_1, x_3\}$	B	3	0	3	0
$\{x_2, x_4\}$	E	0	3	0	3
$\{x_3, x_4\}$	F	0	0	6	6



A three-dimensional example

$$
\begin{array}{rlcccccccccccc}
\min & \text{whatever} \\
\text{s.t.} & x_1 & & & + & x_3 & + & x_4 & & & & & = & 2 \\
& & & x_2 & & & & & + & x_5 & & & = & 1 \\
& & & & & x_3 & & & & & + & x_6 & = & 1 \\
& x_i \geq 0 \quad \forall i = 1, ..., 6.
\end{array}
$$

Basis	Point	Basic solution		
		x_1	x_2	x_3
$\{x_4, x_5, x_6\}$	A	0	0	0
$\{x_1, x_5, x_6\}$	B	2	0	0
$\{x_1, x_2, x_6\}$	C	2	1	0
$\{x_2, x_4, x_6\}$	D	0	1	0
$\{x_3, x_4, x_5\}$	E	0	0	1
$\{x_1, x_3, x_5\}$	F	1	0	1
$\{x_1, x_2, x_3\}$	G	1	1	1
$\{x_2, x_3, x_4\}$	H	0	1	1



A better way to search

- Given all these concepts, how would you search among bfs?
- At each bfs, move to an **adjacent** bfs that is **better**!
  - Around the current bfs, there should be some improving directions.
  - Otherwise, the bfs is optimal.
- Next we will introduce the simplex method, which utilize this idea in an elegant way.

Road map

- Standard form LPs.
- Basic solutions.
- Basic feasible solutions.
- **The geometry of the simplex method**.
- The algebra of the simplex method.

The simplex method

- All we need is to search among bfs.
  - Geometrically, we search among extreme points.
  - Moving to an adjacent bfs is to move along an edge.
- Questions:
  - Which edge to move along?
  - When to stop moving?
- All these must be done with algebra rather than geometry.
  - Nevertheless, geometry provides intuitions.
- Algebraically, to move to an adjacent bfs, we need to **replace** one basic variable by a nonbasic variable.
  - E.g., moving from $B_1 = \{x_1, x_2, x_3\}$ to $B_2 = \{x_2, x_3, x_5\}$.
- There are two things to do:
  - Select one **nonbasic** variable to **enter** the basis, and
  - Select one **basic** variable to **leave** the basis.

The entering variable

- Selecting one nonbasic variable to enter means making it **nonzero**.
  - One constraint becomes **nonbinding**.
  - We move along the edge that moves **away from** the constraint.
- We will illustrate this idea with the following LP

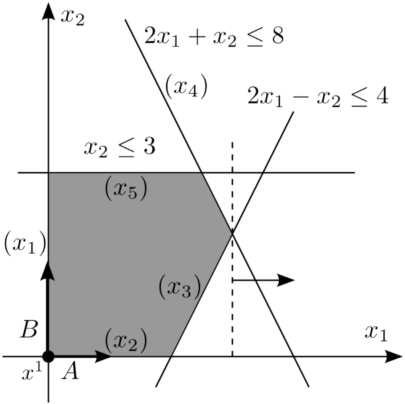
$$
\begin{array}{rrcrcl}
\min & -x_1 & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\
& 2x_1 & + & x_2 & \leq & 8 \\
& & & x_2 & \leq & 3 \\
& \multicolumn{5}{l}{x_i \geq 0 \quad \forall\ i = 1, 2.}
\end{array}
$$

and its standard form

$$
\begin{array}{rrcrcrcrcrcl}
\min & -x_1 & & & & & & & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\
& 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\
& & & x_2 & & & & & + & x_5 & = & 3 \\
& \multicolumn{11}{l}{x_i \geq 0 \quad \forall\ i = 1, ..., 5.}
\end{array}
$$

The entering variable

- For the bfs $x^1 = (0, 0, 4, 8, 3)$:
  - The basis is $\{x_3, x_4, x_5\}$.
  - x_1 and x_2 are nonbasic.
  - x_1 and x_2 may enter the basis.
  - Letting x_1 enters
    $\Rightarrow$ making $x_1 > 0$
    $\Rightarrow$ moving away from $x_1 \geq 0$
    $\Rightarrow$ moving along direction A.
  - Letting x_2 enters
    $\Rightarrow$ making $x_2 > 0$
    $\Rightarrow$ moving away from $x_2 \geq 0$
    $\Rightarrow$ moving along direction B.

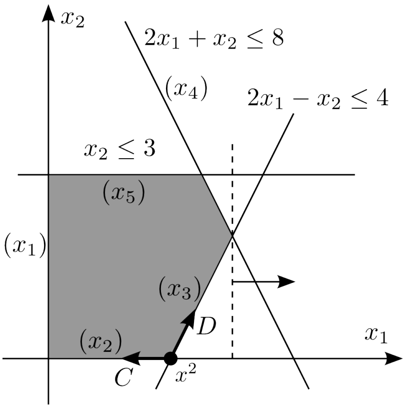


The entering variable

- For the bfs $x^2 = (2, 0, 0, 4, 3)$:
  - The basis is $\{x_1, x_4, x_5\}$.
  - x_2 and x_3 are nonbasic.
  - x_2 and x_3 may enter the basis.
  - Letting x_2 enters
    $\Rightarrow$ making $x_2 > 0$
    $\Rightarrow$ moving away from $x_2 \geq 0$
    $\Rightarrow$ moving along direction D.
  - Letting x_3 enters
    $\Rightarrow$ making $x_3 > 0$
    $\Rightarrow$ moving away from

$$2x_1 - x_2 + x_3 = 4$$

    $\Rightarrow$ moving along direction C.

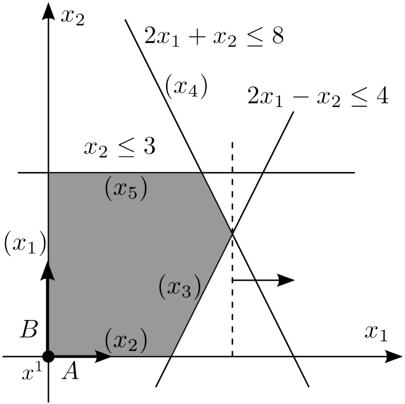


The leaving variable

- Suppose we have chosen one entering variable.
  - We have chosen one edge to move along.
- How to choose a **leaving** variable?
  - When should we **stop**?
- Geometrically, we stop when we "**hit a constraint**".
  - We are moving along edges, so all equalities constraints will remain to be satisfied. Only nonnegativity constraints may be violated.
- Albegraically, we stop when one basic variable **decreases to 0**.
  - This basic variable will leave the basis.
  - As it becomes 0, it becomes a nonbasic variable.

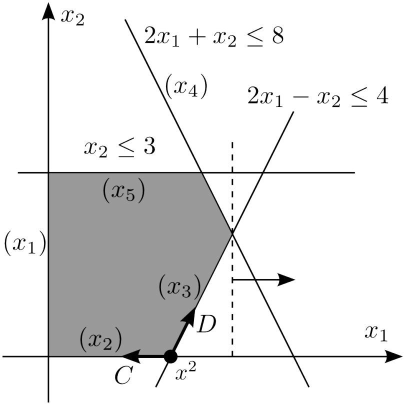
The leaving variable

- For the bfs x^1, suppose we move along direction A.
  - The original basis is $\{x_3, x_4, x_5\}$.
  - x_1 **enters** the basis.
  - We **first hit** $2x_1 - x_2 \leq 4$.
  - x_3 becomes 0.
  - x_3 becomes nonbasic.
  - x_3 **leaves** the basis.
  - The new basis is $\{x_1, x_4, x_5\}$.



The leaving variable

- For the bfs x^2, suppose we move along direction D.
  - The original basis is $\{x_1, x_4, x_5\}$.
  - x_2 enters the basis.
  - We first hit $2x_1 + x_2 \leq 8$.
  - x_4 becomes 0.
  - x_4 becomes nonbasic.
  - x_4 **leaves** the basis.
  - The new basis is $\{x_1, x_2, x_5\}$.



An iteration

- At a bfs, we move to another **better** bfs.
  - We first choose **which direction to go** (the **entering** variable). That should be an improving direction along an edge.
  - We then determine **when to stop** (the **leaving** variable). That depends on the first constraint we hit.
  - We may then treat the new bfs as the current bfs and then **repeat**.
- We stop when there is no improving direction.
- The process of moving to the next bfs is call an **iteration**.

The simplex method

- The simplex method is simple:
  - It suffices to **move along edges** (because we only need to search among extreme points).
  - At each point, the number of directions to search for is **small** (because we consider only edges).
  - For each improving direction, the **stopping condition** is simple: Keep moving forwards until we cannot.
- The simplex method is smart:
  - When at a point there is **no improving direction** along an edge, the point is optimal.
- Next let's know exactly how to run the simplex method in algebra.

Road map

- Standard form LPs.
- Basic solutions.
- Basic feasible solutions.
- The geometry of the simplex method.
- **The algebra of the simplex method**.

The simplex method

- To introduce the algebra of the simplex method, let's consider the following LP

$$\begin{array}{rrcrcl} \min & -2x_1 & - & 3x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 6 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & & & \multicolumn{3}{l}{x_i \geq 0 \quad \forall\, i = 1, 2} \end{array}$$

and its standard form

$$\begin{array}{rrcrcrcrcl} \min & -2x_1 & - & 3x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & 2x_1 & + & x_2 & & & + & x_4 & = & 8 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall\, i = 1, ..., 4.} \end{array}$$

System of equalities

- We need to keep track of the **objective value**.
  - We want to keep improving our solution.
  - We will use $z = -2x_1 - 3x_2$ to denote the objective value.
  - The objective value will sometimes be called **the z value**.
- Once we keep in mind that (1) we are minimizing z and (2) all variables (except z) must be nonnegativie, the standard form is nothing but a system of three equalities:

$$
\begin{array}{ccccccccccc}
z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

  - Note that $z = -2x_1 - 3x_2$ is expressed as $z + 2x_1 + 3x_2 = 0$.
  - This "constraint" (which actually represents the objective function) will be called the 0th constraint.
- We will repeatedly use Linear Algebra to solve the system.

An initial bfs

- To start, we need to first have an **initial bfs**.
  - For this example, a basis is a set of **two** variables such that A_B, the matrix formed by the two corresponding columns, is invertible.
  - By satisfying $A_B x_B = b$, a bfs has all its basic variables x_B nonnegative.
  - How may we get one bfs?
- Investigate the system in details:

$$
\begin{array}{ccccccccccc}
z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

  - Selecting x_3 and x_4 definitely works!
  - In the system, these two columns form an **identity matrix**: $A_B = I$.[7]
  - Moreover, in a standard form LP, the RHS b are nonnegative.
  - Therefore, $x_B = A_B^{-1} b = Ib = b \geq 0$.

[7] For what kind of LPs does this identity matrix exist?

Improving the current bfs

$$
\begin{array}{ccccccccccc}
z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

- Let us start from $x^1 = (0,0,6,8)$ and $z_1 = 0$.
- To move, let's choose a nonbasic variable to enter. x_1 or x_2?
  - The **0th constraints** tells us that entering either variable makes z smaller: When one goes up, z goes down to maintain the equality.
  - For no reason, let's choose x_1 to enter.
- When to stop?
  - Now x_1 goes up from 0.
  - $(0,0,6,8) \to (1,0,5,6) \to (2,0,4,4) \to \cdots$. Note that x_2 remains 0.
  - We will stop at $(4,0,2,0)$, i.e., when x_4 becomes 0.
  - This is indicated by the **ratio** of the **RHS** and **entering column**: Because $\frac{8}{2} < \frac{6}{1}$, x_4 becomes 0 sooner than x_3.
- We move to $x^2 = (4,0,2,0)$ with $z_2 = -8$.

Keep improving the current bfs

$$\begin{array}{rcrcrcrcrcl} z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. \end{array}$$

- So far so good!
- Let's improve $x^2 = (4, 0, 2, 0)$ by moving to the next bfs.
  - One of x_2 and x_4 may enter.
- According to the 0th row, we should let x_2 enter.[8]
- When x_2 goes up and x_4 remains 0:
  - The 2nd row says x_2 can at most become 8 (and then x_1 becomes 0).
  - In the 1st row... how will x_1 and x_3 change???????
- An easier way is to **update the system** before the 2nd move.
  - So that in each row there is **only one** basic variable.
- Let's see how to update the system **every time** when we make a move.

[8] This statement is actually wrong. Why?

Rewriting the standard form

- Recall that a standard form LP is

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array}$$

- Given a basis B, we may split x into (x_B, x_N).
- We may also split c into (c_B, c_N) and A into (A_B, A_N).
  - $c_B \in \mathbb{R}^m$, $c_N \in \mathbb{R}^{n-m}$, $A_B \in \mathbb{R}^{m \times m}$, and $A_N \in \mathbb{R}^{m \times (n-m)}$.
- With the splits, the LP becomes

$$\begin{array}{rl} \min & c_B^T x_B + c_N^T x_N \\ \text{s.t.} & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0. \end{array} \quad \text{or} \quad \begin{array}{rl} \min & c_B^T \left[A_B^{-1}(b - A_N x_N) \right] + c_N^T x_N \\ \text{s.t.} & x_B = A_B^{-1}(b - A_N x_N) \\ & x_B, x_N \geq 0. \end{array}$$

Rewriting the standard form

- With some more algebra, the LP becomes

$$\begin{aligned} \min \quad & c_B^T A_B^{-1} b - \left(c_B^T A_B^{-1} A_N - c_N^T \right) x_N \\ \text{s.t.} \quad & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ & x_B, x_N \geq 0. \end{aligned}$$

- By expressing the objective function by an equation with z, the LP can be expressed as

$$\begin{array}{ccccccc} z & & + & \left(c_B^T A_B^{-1} A_N - c_N \right) x_N & = & c_B^T A_B^{-1} b & \text{(0th row)} \\ & I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b. & \text{(1st to } m\text{th row)} \end{array}$$

Rewriting the standard form

- What are we doing?
- Given a basis B, we update the system to make two things happen at the **basic columns**:
  - There is an identity matrix at the 1st to mth row:

$$\begin{array}{lcccccl} z & & + & (c_B^T A_B^{-1} A_N - c_N^T) x_N & = & c_B^T A_B^{-1} b & \text{(0th row)} \\ & \boxed{I x_B} & + & A_B^{-1} A_N x_N & = & A_B^{-1} b. & \text{(1st to } m\text{th row)} \end{array}$$

  - All numbers are zero at the 0th row:

$$\begin{array}{lcccccl} z & \boxed{} & + & (c_B^T A_B^{-1} A_N - c_N^T) x_N & = & c_B^T A_B^{-1} b & \text{(0th row)} \\ & I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b. & \text{(1st to } m\text{th row)} \end{array}$$

- Then we know what will happen when a nonbasic variable enters!

Improving the current bfs (the 2nd attempt)

- Recall that for the system

$$
\begin{array}{ccccccccccc}
z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8,
\end{array}
$$

  we start from $x^1 = (0, 0, 6, 8)$ with $z_1 = 0$.
  - For the basic columns (the 3rd and 4th ones), indeed we have the identity matrix and zeros.
- Then we know x_1 enters and x_4 leaves.
  - The basis becomes $\{x_1, x_3\}$.
  - We need to update the system to

$$
\begin{array}{ccccccccccc}
z & + & & + & ?x_2 & & & + & ?x_4 & = & 0 \\
 & & & + & ?x_2 & + & x_3 & + & ?x_4 & = & 6 \\
 & & x_1 & + & ?x_2 & & & + & ?x_4 & = & 8.
\end{array}
$$

  - How? **Elementary row operations**!

Updating the system

- Starting from:

$$
\begin{array}{rcrcrcrcrcrl}
z & + & 2x_1 & + & 3x_2 & & & & & = & 0 & (0) \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 & (1) \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. & (2)
\end{array}
$$

  - Multiply (2) by $\frac{1}{2}$: $x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 = 4$.
  - Multiply (2) by -1 and then add it into (1): $\frac{3}{2}x_2 + x_3 - \frac{1}{2}x_4 = 2$.
  - Multiply (2) by -1 and then add it into (0): $z + 2x_2 - x_4 = -8$.
- Collectively, the system becomes

$$
\begin{array}{rcrcrcrcrcrl}
z & & & + & 2x_2 & & & - & x_4 & = & -8 & (0) \\
 & & & + & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 & (1) \\
 & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4. & (2)
\end{array}
$$

Improving the current bfs (finally!)

- Given the updated system

$$\begin{array}{rcrcrcrcrcrl} z & & & + & 2x_2 & & & - & x_4 & = & -8 & (0) \\ & & & + & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 & (1) \\ & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4, & (2) \end{array}$$

  we now know how to do the next iteration.
  - We are at $x^2 = (4, 0, 2, 0)$ with $z_2 = -8$.
  - One of x_2 and x_4 may enter.
  - If x_2 enters, z will go down. Good!
  - If x_4 enters, z will go up. Bad.
- Let x_2 enter:
  - Row 1: When x_2 goes up, x_3 goes down. x_2 can be as large as $\frac{2}{3/2} = \frac{4}{3}$.
  - Row 2: When x_2 goes up, x_1 goes down. x_2 can be as large as $\frac{4}{1/2} = 8$.
  - So x_3 becomes 0 sooner than x_1. x_3 leaves the basis.
- The basic variables become x_1 and x_2. Let's update again.

Improving once more

- Given the system

$$\begin{array}{rcrcrcrcrcl} z & & & + & 2x_2 & & & - & x_4 & = & -8 \quad (0) \\ & & & + & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 \quad (1) \\ & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4, \quad (2) \end{array}$$

  we now need to update it to fit the new basis $\{x_1, x_2\}$.
  - Multiply (1) by $\frac{2}{3}$: $x_2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{4}{3}$.
  - Multiply (the updated) (1) by $-\frac{1}{2}$ and add it to (2).
  - Multiply (the updated) (1) by -2 and add it to (0).
- We get

$$\begin{array}{rcrcrcrcl} z & & & - & \frac{4}{3}x_3 & - & \frac{1}{3}x_4 & = & -\frac{32}{3} \quad (0) \\ & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} \quad (1) \\ & x_1 & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3}. \quad (2) \end{array}$$

No more improvement!

- The system

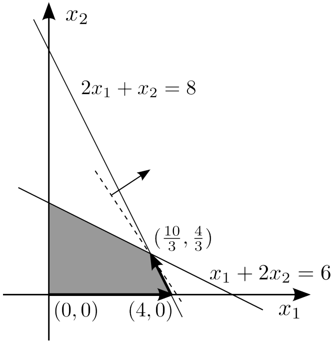
$$
\begin{array}{rcrcrcrcrl}
z & & & - & \frac{4}{3}x_3 & - & \frac{1}{3}x_4 & = & -\frac{32}{3} & (0) \\
 & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} & (1) \\
x_1 & & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3} & (2)
\end{array}
$$

  tells us that the new bfs is $x^3 = (\frac{10}{3}, \frac{4}{3}, 0, 0)$ with $z_3 = -\frac{32}{3}$.
  - Updating the system also gives us the new bfs and its objective value.
- Now... no more improvement is needed!
  - Entering x_3 makes things worse (z must go up).
  - Entering x_4 also makes things worse.
- x^3 is an optimal solution.[9] We are done!

[9] This is indeed true, though a rigorous proof is omitted.

Visualizing the iterations

- Let's visualize this example and relate bfs with extreme points.
  - The initial bfs corresponds to $(0, 0)$.
  - After one iteration, we move to $(4, 0)$.
  - After two iterations, we move to $(\frac{10}{3}, \frac{4}{3})$, which is optimal.
- Please note that we move along edges to search among extreme points!



Summary

- To run the simplex method:
  - Find an initial bfs with its basis.[10]
  - Among those nonbasic variables with positive coefficients in the 0th row, choose one to enter.[11]
    - If there is none, terminate and report the current bfs as optimal.
  - According to the ratios from the basic and RHS columns, decide which basic variable should leave.[12]
  - Find a new basis.
  - Make the system fit the requirements for basic columns:
    - Identity matrix in constraints (1st to mth row).
    - Zeros in the objective function (0th row).
  - Repeat.

[10] How to find one?

[11] What if there are multiple?

[12] What if there is a tie? What if the denominator is 0 or negative?

The tableau representation

- Just as what we did for Gaussian eliminations, we typically omit variables when updating those systems.
- We organize coefficients into **tableaus**.
  - As the column with z never changes, we do not include it in a tableau.
- For our example, the initial system

$$\begin{array}{ccccccccccc} z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. \end{array}$$

can be expressed as

$$\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 0 & x_3 = 6 \\ 2 & 1 & 0 & 1 & x_4 = 8 \end{array}$$

- The basic columns have zeros in the 0th row and an identity matrix in the other rows.
- The identity matrix associates each row with a basic variable.
- A posivie number in the 0th row of a nonbasic column means that variable can enter.

Using tableaus rather than systems

$$\begin{array}{rcrcrcrcrcr} z & + & 2x_1 & + & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8 \end{array}$$

$$\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 0 & x_3 = 6 \\ \boxed{2} & 1 & 0 & 1 & x_4 = 8 \end{array}$$

$$\downarrow$$

$$\begin{array}{rcrcrcrcrcr} z & & & + & 2x_2 & & & - & x_4 & = & -8 \\ & & & + & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 \\ & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4 \end{array}$$

$$\begin{array}{cccc|c} 0 & 2 & 0 & -1 & -8 \\ \hline 0 & \boxed{\frac{3}{2}} & 1 & -\frac{1}{2} & x_3 = 2 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & x_1 = 4 \end{array}$$

$$\downarrow$$

$$\begin{array}{rcrcrcrcrcr} z & & & & & - & \frac{4}{3}x_3 & - & \frac{1}{3}x_4 & = & -\frac{32}{3} \\ & & & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} \\ & & x_1 & & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3} \end{array}$$

$$\begin{array}{cccc|c} 0 & 0 & -\frac{4}{3} & -\frac{1}{3} & -\frac{32}{3} \\ \hline 0 & 1 & \frac{2}{3} & -\frac{1}{3} & x_2 = \frac{4}{3} \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & x_1 = \frac{10}{3} \end{array}$$

The second example

- Consider another example:

$$
\begin{array}{rrcrcl}
\max & x_1 & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\
 & 2x_1 & + & x_2 & \leq & 8 \\
 & & & x_2 & \leq & 3 \\
 & \multicolumn{5}{l}{x_i \geq 0 \quad \forall\ i = 1, 2.}
\end{array}
$$

- The standard form is

$$
\begin{array}{rrcrcrcrcrcl}
\max & x_1 & & & & & & & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\
 & & & x_2 & & & & & + & x_5 & = & 3 \\
 & \multicolumn{11}{l}{x_i \geq 0 \quad \forall\ i = 1, ..., 5.}
\end{array}
$$

The first iteration

- We prepare the initial tableau. We have $x^1 = (0, 0, 4, 8, 3)$ and $z_1 = 0$.

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
$$

- For this **maximization** problem, we look for **negative** numbers in the 0th row. Therefore, x_1 enters.
  - Those numbers in the 0th row are called **reduced costs**.
  - The 0th row is $z - x_1 = 0$. Increasing x_1 can increase z.
- "Dividing the RHS column by the entering column" tells us that x_3 should leave (it has the minimum ratio).[13]
  - This is called the **ratio test**. We **always** look for the smallest ratio.

[13] The 0 in the 3rd row means that increasing x_1 does not affect x_5.

The first iteration

- x_1 enters and x_3 leaves. The next tableau is found by **pivoting** at 2:

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\boxed{2} & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & 2 & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
$$

- The new bfs is $x^2 = (2, 0, 0, 4, 3)$ with $z_2 = 2$.
- Continue?
  - There is a negative reduced cost in the 2nd column: x_2 enters.
- Ratio test:
  - That $-\frac{1}{2}$ in the 1st row shows that increasing x_2 makes x_1 larger. Row 1 does not participate in the ratio test.
  - For rows 2 and 3, row 2 wins (with a smaller ratio).

The second iteration

- x_2 enters and x_4 leaves. We pivot at 2.
- The second iteration is

$$
\begin{array}{ccccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & \boxed{2} & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 3 \\
\hline
1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & x_1 = 3 \\
0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 & x_2 = 2 \\
0 & 0 & \frac{1}{2} & \frac{-1}{2} & 1 & x_5 = 1
\end{array}
$$

- The third bfs is $x^3 = (3, 2, 0, 0, 1)$ with $z_3 = 3$.
  - It is optimal (why?).
  - Typically we write the optimal solution we find as x^* and optimal objective value as z^*.

Verifying our solution

- The three basic feasible solutions we obtain are
  - $x^1 = (0, 0, 4, 8, 3)$.
  - $x^2 = (2, 0, 0, 4, 3)$.
  - $x^3 = x^* = (3, 2, 0, 0, 1)$.

  Do they fit our graphical approach?