Operations Research

Applications of Linear Programming

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Road map

- **Materials blending**.
- Linearizing maximum/minimum functions.
- AMPL.

Material blending

- In some situations, we need to determine not only products to produce but also **materials** to input.[1]
- This is because we have some **flexibility** in making the products.
- For example, in making orange juice, we may use orange, sugar, water, etc. Different ways of **blending** these materials results in different qualities of juice.
- The goal is to save money (lower the proportion of expensive materials) while maintaining **quality**.

[1] This example comes from Chapter 3 of *Operations Research: Applications and Algorithms* by Wayne L. Winston, 4th edition.

Material blending: the problem

- We blend materials 1, 2, and 3 to make products 1 and 2.
- The quality of a product, which depends on the proportions of these three materials, must meet the standard:
  - Product 1: at least 40% of material 1; at least 20% of material 2.
  - Product 2: at least 50% of material 1; at most 30% of material 3.
- At most 100 kg of product 1 and 150 kg of product 2 can be sold.
- Prices for products 1 and 2 are \$10 and \$15 per kg, respectively.
- Costs for materials 1 to 3 are \$8, \$4, and \$3 per kg, respectively.
- Amount of a product made equals the amount of materials input.
- We want to maximize the total profit.

Formulation: decision variables

- Probably our first attempt is to define the following: Let

$$
\begin{aligned}
x_1 &= \text{kg of product 1 produced,}\\
x_2 &= \text{kg of product 2 produced,}\\
y_1 &= \text{kg of material 1 purchased,}\\
y_2 &= \text{kg of material 2 purchased, and}\\
y_3 &= \text{kg of material 3 purchased.}
\end{aligned}
$$

- May we express the quality of each product? No!
- We need to specify the amount of material 1 used for product 1, the amount of material 1 used for product 2, etc.
- So we need to **redefine** our decision variables.

Formulation: decision variables

- How about this: Let

$$
\begin{aligned}
x_1 &= \text{kg of material 1 used for product 1,} \\
x_2 &= \text{kg of material 1 used for product 2,} \\
x_3 &= \text{kg of material 2 used for product 1,} \\
x_4 &= \text{kg of material 2 used for product 2,} \\
x_5 &= \text{kg of material 3 used for product 1, and} \\
x_6 &= \text{kg of material 3 used for product 2.}
\end{aligned}
$$

- The definition is correct and precise, but **not easy to use**.
  - Similar to computer programming: give your variables reasonable names that allow people to know **what they are**.

Formulation: decision variables

- A more intuitive way of naming variables: Let

$$
\begin{aligned}
x_{11} &= \text{kg of material 1 used for product 1,} \\
x_{12} &= \text{kg of material 1 used for product 2,} \\
x_{21} &= \text{kg of material 2 used for product 1,} \\
x_{22} &= \text{kg of material 2 used for product 2,} \\
x_{31} &= \text{kg of material 3 used for product 1, and} \\
x_{32} &= \text{kg of material 3 used for product 2.}
\end{aligned}
$$

- Or in a compact format:

$$x_{ij} = \text{kg of material } i \text{ used for product } j, i = 1, ..., 3, j = 1, 2.$$

Formulation: objective function

- Let's write down the total profit.
- Sales revenues depend on the amount of products we sell.
  - How many kg of product 1 may we sell? $x_{11} + x_{21} + x_{31}$ kg.
  - Similarly, we have $x_{12} + x_{22} + x_{32}$ kg of product 2.
- Material costs depend on the amount of materials we purchase.
  - Similarly, we need to buy $x_{11} + x_{12}$ kg of material 1, $x_{21} + x_{22}$ kg of material 2 and $x_{31} + x_{32}$ kg of material 3.
- The objective function is

$$\begin{aligned} &\max\ 10(x_{11} + x_{21} + x_{31}) + 15(x_{12} + x_{22} + x_{32}) \\ &\qquad - 8(x_{11} + x_{12}) - 4(x_{21} + x_{22}) - 3(x_{31} + x_{32}) \\ = &\max\ 2x_{11} + 7x_{12} + 6x_{21} + 11x_{22} + 7x_{31} + 12x_{32}. \end{aligned}$$

Formulation: quality constraints

- To guarantee that at least 40% of product 1 are made by material 1?

$$\frac{x_{11}}{x_{11}+x_{21}+x_{31}} \geq 0.4.$$

- It is conceptually correct. However, it is **nonlinear**!
- Let's fix the nonlinearity by moving the denominator to the RHS:

$$x_{11} \geq 0.4(x_{11}+x_{21}+x_{31}).$$

  Though equivalent, they are just different.
- We may (but are not required to) choose other format, such as

$$0.6x_{11} - 0.4x_{21} - 0.4x_{31} \geq 0 \quad \text{or} \quad 3x_{11} - 2x_{21} - 2x_{31} \geq 0.$$

Formulation: constraints

- In total we have four quality constraints:
  - $x_{11} \geq 0.4(x_{11} + x_{21} + x_{31})$.
  - $x_{21} \geq 0.2(x_{11} + x_{21} + x_{31})$.
  - $x_{12} \geq 0.5(x_{12} + x_{22} + x_{32})$.
  - $x_{13} \leq 0.3(x_{12} + x_{22} + x_{32})$.
- The demands are limited:

$$x_{11} + x_{21} + x_{31} \leq 100 \quad \text{and} \quad x_{12} + x_{22} + x_{32} \leq 150.$$

- The quantities are nonnegative:

$$x_{ij} \geq 0 \quad \forall i = 1, ..., 3, j = 1, 2.$$

Formulation: the complete formulation

- The complete formulation is

$$
\begin{array}{rl}
\max & 10(x_{11}+x_{21}+x_{31})+15(x_{12}+x_{22}+x_{32}) \\
& -8(x_{11}+x_{12})-4(x_{21}+x_{22})-3(x_{31}+x_{32}) \\
\text{s.t.} & x_{11} \geq 0.4(x_{11}+x_{21}+x_{31}), \quad x_{21} \geq 0.2(x_{11}+x_{21}+x_{31}), \\
& x_{12} \geq 0.5(x_{12}+x_{22}+x_{32}) \quad x_{13} \leq 0.3(x_{12}+x_{22}+x_{32}) \\
& x_{11}+x_{21}+x_{31} \leq 100, \quad x_{12}+x_{22}+x_{32} \leq 150 \\
& x_{ij} \geq 0 \quad \forall i=1,...,3, j=1,2.
\end{array}
$$

- Some remarks:
  - We may need to redefine decision variables when it is necessary.
  - We may from time to time use multi-dimensional variables.
  - We need to **linearize** nonlinear constraints or objective functions, even if they look so similar.

Road map

- Materials blending.
- **Linearizing maximum/minimum functions**.
- AMPL.

Fair allocation: the problem

- Suppose that we want to allocate \$1000 to two persons in a **fair** way.
- We adopt the following measurement of fairness: The smaller the difference between the two amounts, the fairer the allocation is.
- Obviously the answer is to give each person \$500.
- May we formulate a linear program to solve this problem?

Fair allocation: the first attempt

- Let x_i be the amount allocated to person i, $i = 1, 2$.
- Is the following formulation correct?

$$\begin{aligned} \min \quad & x_2 - x_1 \\ \text{s.t.} \quad & x_1 + x_2 = 1000 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

Fair allocation: the second attempt

- Let x_i be the amount allocated to person i, $i = 1, 2$.
- The following formulation is correct:

$$\begin{aligned} \min \quad & |x_2 - x_1| \\ \text{s.t.} \quad & x_1 + x_2 = 1000 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- However, the absolute function $|\cdot|$ is **nonlinear**!
- It is possible to linearize this problem as a linear program?

Linearizing the second attempt

- First, let w be the absolute difference: $w = |x_2 - x_1|$:

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w = |x_2 - x_1| \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- We may change this equality constraint to an inequality:

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w \geq |x_2 - x_1| \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

Why?

Linearizing the second attempt

- Now, notice that $|x_2 - x_1| = \max\{x_2 - x_1, x_1 - x_2\}$ and

$$w \geq \max\{x_2 - x_1, x_1 - x_2\} \quad \Leftrightarrow \quad w \geq x_2 - x_1 \text{ and } w \geq x_1 - x_2.$$

- Therefore, the linear program we want is

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w \geq x_2 - x_1 \\ & w \geq x_1 - x_2 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- May we solve this LP and get the $(500, 500)$ allocation?

Solving the linear program

- Consider the LP

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w \geq x_2 - x_1 \\ & w \geq x_1 - x_2 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- The equality constraint means that $x_2 = 1000 - x_1$:

$$\begin{array}{rl} \min & w \\ \text{s.t.} & w \geq 1000 - 2x_1 \\ & w \geq 2x_1 - 1000 \\ & x_1 \geq 0. \end{array}$$

- Would you graphically solve the LP?

Linearizing constraints

- The technique we just applied can be generalized.
- When a **maximum** function is at the **smaller** side of an inequality:

$$y \geq \max\{x_1, x_2\} \quad \Leftrightarrow \quad y \geq x_1 \text{ and } y \geq x_2.$$

- y, x_1, and x_2 can be variables, parameters, or a function of them:

$$\begin{aligned} & y + x_1 + 3 \geq \max\{x_1 - x_3, 2x_2 + 4\} \\ \Leftrightarrow \quad & y + x_1 + 3 \geq x_1 - x_3 \text{ and } y + x_1 + 3 \geq 2x_2 + 4. \end{aligned}$$

- There may be more than two terms in the maximum function:

$$y \geq \max_{i=1,\ldots,n}\{x_i\} \quad \Leftrightarrow \quad y \geq x_i \quad \forall i = 1, ..., n.$$

Linearizing constraints

- A **minimum** function at the **larger** side can also be linearized.

$$\begin{aligned} & y + x_1 \leq \min\{x_1 - x_3, 2x_2 + 4, 0\} \\ \Leftrightarrow \quad & y + x_1 \leq x_1 - x_3,\ y + x_1 \leq 2x_2 + 4,\ \text{and } y + x_1 \leq 0. \end{aligned}$$

- This technique **does not** apply to:
  - A maximum function at the larger side: $y \leq \max\{x_1, x_2\}$ is not equivalent to $y \leq x_1$ and $y \leq x_2$.
  - A minimum function at the smaller side: $y \geq \min\{x_1, x_2\}$ is not equivalent to $y \geq x_1$ and $y \geq x_2$.
  - A maximum or minimum function in an equality.

Linearizing constraints

- A **minimum** function at the **larger** side can also be linearized.

$$
\begin{aligned}
& y + x_1 \leq \min\{x_1 - x_3, 2x_2 + 4, 0\} \\
\Leftrightarrow \quad & y + x_1 \leq x_1 - x_3,\ y + x_1 \leq 2x_2 + 4,\ \text{and } y + x_1 \leq 0.
\end{aligned}
$$

- This technique **does not** apply to:
  - A maximum function at the larger side: $y \leq \max\{x_1, x_2\}$ is not equivalent to $y \leq x_1$ and $y \leq x_2$.
  - A minimum function at the smaller side: $y \geq \min\{x_1, x_2\}$ is not equivalent to $y \geq x_1$ and $y \geq x_2$.
  - A maximum or minimum function in an equality.

Linearizing the objective function

- When we **minimize a maximum function**:

$$\min \ \max\{x_1, x_2\} \quad \Leftrightarrow \quad \begin{array}{rl} \min & w \\ \text{s.t.} & w \geq x_1 \\ & w \geq x_2. \end{array}$$

  - x_1 and x_2 can be variables, parameters, or a function of them.
  - There may be other constraints.
  - The objective function may contain other terms.
- Similarly, when we **maximize a minimum function**:

$$\begin{array}{rl} \max & \min\{x_1, x_2, 2x_3 + 5\} + x_4 \\ \text{s.t.} & 2x_1 + x_2 - x_4 \leq x_3. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \max & w + x_4 \\ \text{s.t.} & w \leq x_1 \\ & w \leq x_2 \\ & w \leq 2x_3 + 5 \\ & 2x_1 + x_2 - x_4 \leq x_3. \end{array}$$

Linearizing the objective function

- This technique does not apply to:
  - Maximizing a maximum function.
  - Minimizing a minimum function.
- Finally, an **absolute function** is just a maximum function:

$$|x| = \max\{x, -x\}.$$

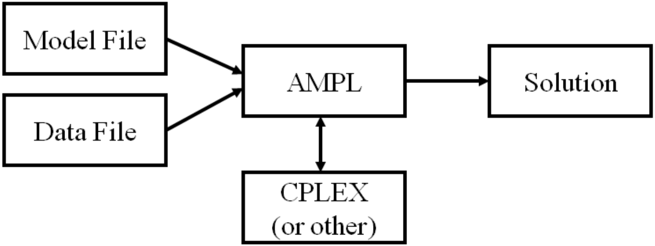
  - Minimizing an absolute function can be linearized.
  - An absolute function at the smaller side of an inequality can be linearized.

Road map

- Materials blending.
- Linearizing maximum/minimum functions.
- **AMPL**.

AMPL

- **AMPL** means "A Modeling Language for Mathematical Programming."
- AMPL is an interface, and **CPLEX** is a solver.



To obtain AMPL

- Office website: `http://ampl.com/`.
- To download a size-limited student version: `http://ampl.com/try-ampl/download-a-demo-version/`.
- A typical package includes the components:
  - **`ampl`**: The console environment.
  - **`cplex`**: A solver for linear (fractional or integer) programs.
  - **`minos`**: A solver for nonlinear (fractional) programs.
  - **`sw`**: A more user-friendly console environment (MS Windows only) for "scrolling windows."
- In this course, we use the licensed full education version.

The first example

- Consider our favorite LP

$$
\begin{aligned}
z^* = \max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 6 \\
& x_i \geq 0 \quad \forall i = 1, 2.
\end{aligned}
$$

- An optimal solution is $x^* = (2, 2)$. The associated $z^* = 6$.

The first example

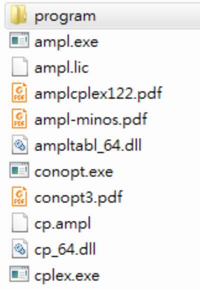
- To use AMPL to solve this LP, all we need is a **model file**:

```
var x1;
var x2;

maximize profit: x1 + x2;

subject to resource_1: x1 + 2 * x2 <= 6;
subject to resource_2: 2 * x1 + x2 <= 6;
subject to nonneg_1: x1 >= 0;
subject to nonneg_2: x2 >= 0;
```

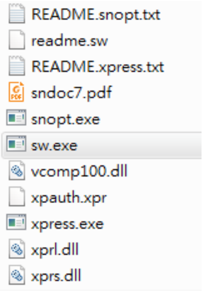
- Let's put these codes into a plain text file called "eg1.mod" and save this file in a "program" (or other name you prefer) folder.
- Let's try it first and explain the codes later.

The first example

Put your "eg1.mod" in the "program" folder.



Open the console environment `sw` (or `ampl` in Mac).

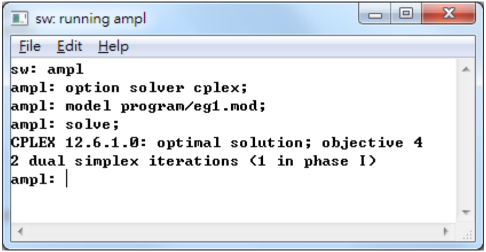


The first example

- Type the following instructions one by one:

```
ampl
option solver cplex;
model program/eg1.mod;
solve;
```

- An optimal solution is found!
  - With the solver CPLEX.
  - The objective value of the optimal solution is 4.

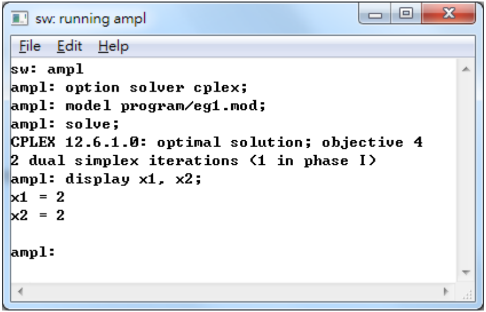


The first example

- To see the optimal solution, type

  `display x1, x2;`

- The values are displayed.
  - $x^* = (2, 2)$.



The first example: codes revisited

- Let's explain the codes in the model file.

```
var x1; # use "var" to declare variables
var x2; # each AMPL statement ends with a semicolon

maximize profit: x1 + x2; # name your objective function

subject to resource_1: x1 + 2 * x2 <= 6; # name each constraint
subject to resource_2: 2 * x1 + x2 <= 6;
subject to nonneg_1: x1 >= 0;
subject to nonneg_2: x2 >= 0;
```

  - **Reserved words**: `var`, `maximize`, `minimize`, and `subject to`.
  - Give all constraints and the objective function **distinct names**.
  - Do not forget **colons and semicolons**.
  - Use `#` to write **comments**.

The first example: make modifications

- Let's modify the code (and save the modified file):

```
subject to resource_1: x1 + 3 * x2 <= 6;
```

- Go back to the console and type

```
reset;
model program/eg1.mod;
solve;
display x1, x2;
```

See how the optimal solution changes. Do not forget to **reset**!

- Remarks:
  - The file can be names with any extension file name as long as it is a plain-text file.
  - Be aware of the file path.
  - In Mac, use absolute file paths.

The second example

- Three products, four markets, different production costs and retail prices Find the production and sales plan to maximize profit.

Product	Market 1	Market 2	Market 3	Market 4	Capacity
1	\$20 / \$30	\$40 / \$45	\$15 / \$30	\$30 / \$40	500
2	\$30 / \$35	\$25 / \$30	\$15 / \$35	\$20 / \$30	600
3	\$25 / \$40	\$35 / \$40	\$10 / \$20	\$25 / \$30	400

The mathematical model

- Variables: Let

$$x_{ij} = \text{sales quantity of product } i \text{ at market } j, i = 1, ..., 3, j = 1, ..., 4.$$

- Parameters: We denote the unit cost and price of product i at market j as C_{ij} and P_{ij}, respectively, and the capacity for product i as K_i.
- The mathematical model (an LP):

$$\begin{aligned} \max \quad & \sum_{i=1}^{3}\sum_{j=1}^{4}(P_{ij} - C_{ij})x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{4} x_{ij} \leq K_i \quad \forall i = 1, ..., 3 \\ & x_{ij} \geq 0 \quad \forall i = 1, ..., 3, j = 1, ..., 4. \end{aligned}$$

Decoupling the data from a model

- To make our AMPL programs flexible and extandable, we should **decouple** the data from a model.
- To do this, we will prepare a **model file** and a **data file**.
  - The model file contains a conceptual model.
  - The data file contains the instance parameters.
- They should both be stored as plain-text files. The extension name does not matter.
- Be aware of file paths.
  - Name them as "eg2.mod" and "eg2.dat" and store them in the "program" folder.

The model file

```
param P;                                    # number of product
param M;                                    # number of market

param Capacity{i in 1..P};                  # the capacity vector
param Cost{i in 1..P, j in 1..M};           # the cost matrix
param Price{i in 1..P, j in 1..M};          # the price matrix

var x{i in 1..P, j in 1..M};

maximize profit:                            # use "sum" for summation
  sum{i in 1..P, j in 1..M} (Price[i, j] - Cost[i, j]) * x[i, j];

subject to productCapacity{i in 1..P}:     # constraint indices
  sum{j in 1..M} x[i, j] <= Capacity[i];
subject to nonnegX{i in 1..P, j in 1..M}:
  x[i, j] >= 0;
```

The data file

```
param P := 3;
param M := 4;

param Capacity :=
   1 500
   2 600
   3 400;

param Cost:     1  2  3  4 :=
            1 20 40 15 30
            2 30 25 15 20
            3 25 35 10 25;

param Price:    1  2  3  4 :=
            1 30 45 30 40
            2 35 30 35 30
            3 40 40 20 30;
```

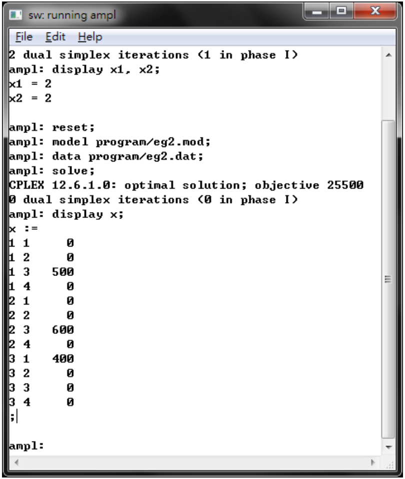
- The format does not matter.
- Reserved words: `param`.
- Parameter names must be consistent with those defined in the model file.
- Array and matrix lengths must be consistent with their limits.
- Be aware of those `:=`, `;`, and `:` and the timing of using them.

Solving the second example

- Solve the second example by loading the model and data files.

```
reset;
model program/eg2.mod;
data program/eg2.dat;
solve;
display x;
```

  - Do not forget to reset!



Some remarks

- The **default solver** in AMPL is MINOS.
  - You may choose to use MINOS by typing `option solver minos`.
  - MINOS can also solve LP.
  - CPLEX uses simplex-based methods while MINOS uses interior search methods (not covered in this course).
  - For solving LPs, CPLEX performs better.
  - MINOS cannot solve integer programs; CPLEX can.
- AMPL is **case-sensitive**.
- Try the AMPL instructions `show;` and `expand;` at home.
- Use `exit;` to exit the AMPL environment.
- The official AMPL book is freely available on the official website.