Operations Research, Spring 2016
Homework 3

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1 Problems

1. (20 points; 5 points each) Consider the function $f(x) = x^3 + ax^2 + bx + c$, where a, b, and c are fixed constants.

   (a) Find the gradient and Hessian of f.
   Hint. While gradients and Hessian matrices are introduced for multi-variate NLPs, they also apply to single-variate NLPs!

   (b) When is the Hessian positive semi-definite?

   (c) Under what condition is f concave over $\mathbb{R}$?

   (d) Suppose the condition obtained in Part (c) is satisfied, find an optimal solution for

   $$\max_{x \in \mathbb{R}} f(x)$$

   as a function of a, b, and c.

2. (20 points; 5 points each) Consider an EOQ problem with annual demand D, unit holding cost h per year, and unit ordering cost K. Suppose we mistakenly believe that the annual demand is $D' = \frac{D}{2}$.

   (a) What is the true EQQ q^* (that minimizes the total cost when annual demand is D)? If we believe that the annual demand is D', what is the wrong order quantity q'?

   (b) Find $\frac{q'}{q^*}$.

   (c) Find $\frac{TC(q')}{TC(q^*)}$, where $TC(q)$ is the annual total cost incurred by ordering q units each time.

   (d) Suppose we mistakenly believe that the annual demand is $D' = kD$, where $k > 0$ is a given constant. Find $\frac{TC(q')}{TC(q^*)}$.

3. (20 points; 5 points each) For each of the following functions, find the region over which the function is convex.

   (a) $f(x_1, x_2) = x_1^2 x_2$.

   (b) $f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2} + x_3$ where the domain is $\{x \in \mathbb{R}^3 | x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.

   (c) $f(x_1, x_2, x_3) = x_1^2 x_2 + 2x_2^2 x_3 + \frac{3}{x_3}$.

   (d) $f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} (x_i + i)^2$.

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4. (20 points; 5 points) A seller is going to sell three products to a market. It will choose the sales quantity for product i as q_i, and then the market clearing price of product i will be $p_i = a_i - b_i q_i$, $i = 1, 2, 3$. The unit procurement cost of product i is c_i. The total number of units across the three products that can be purchased is K. The seller wants to maximize its total profit. For simplicity, let's assume $a_i = a$, $b_i = b$, and $c_i = c$ for all $i = 1, 2, 3$.

   (a) Formulate the seller's problem. Is it a convex program?

   (b) Write down the complete KKT condition for the problem, including primal feasibility, dual feasibility, and complementary slackness.

   (c) Solve this problem by finding an optimal analytical solution.

   (d) How do each of a, b, c, and K affect the optimal solution obtained in Part (c)? Mathematically show it and intuitively explain why.

5. (20 points; 5 points) Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} \; f(x_1, x_2) = e^{x_1} + x_2^2.$$

   (a) Find the gradient and Hessian of f.

   (b) Starting at $(2, 2)$, run one iteration of Newton's method to get to the next solution.

   (c) Starting at $(2, 2)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be $a = 1$.

   (d) Starting at $(2, 2)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be the one that reaches the lowest point along the improving direction. **Hint.** Is this problem unbounded?

2 Submission rules

The deadline of this homework is 12 pm, May 26 (Thursday), 2016. Please either put a hard copy of the work into the instructor's mailbox or submit it to the instructor in class by the due time. Works submitted between 12 pm and 1 pm will get 10 points deducted as a penalty. Submissions later than 1 pm will not be accepted. Each student must submit her/his individual work.