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Suggested Solution for Pre-lecture Problems for Lecture 11

Solution providers: Kiwi Liu and Johnny Chen
Department of Information Management
National Taiwan University

1. (a) Yes.

   (b) No. For example, let the set be F. Let $s = (0, -2)$ and $t = (2, 0)$, therefore $s, t \in F$. Let $\lambda = 0.5$ and the linear combination $\lambda s + (1-\lambda)t = (1, -1) \notin F$. Thus, F is not convex.

   (c) No. For example, let $x_1 = 0, x_2 = -1, \lambda = 0.5$.

   $$f(\lambda x_1 + (1-\lambda)x_2) = f(-0.5) = 1.5 \geq 0.5 = \lambda f(x_1) + (1-\lambda)f(x_2).$$

   Therefore, f is not a convex function.

   (d) Yes.

2. (a)

   $$\frac{\partial f(x)}{\partial x} = f'(x) = 6x + 2$$
   $$\frac{\partial f'(x)}{\partial x} = f''(x) = 6 \geq 0$$

   Therefore, $f(x)$ is a convex function. We may find the global minimum by satisfying FOC. The global minimum is $f'(x) = 0 \Rightarrow x = -\frac{1}{3}$.

   (b)

   $$\frac{\partial f(x)}{\partial x} = f'(x) = 6x^2 - 2x - 2$$
   $$\frac{\partial f'(x)}{\partial x} = f''(x) = 12x - 2$$

   Therefore, for $x \in \left[\frac{1}{6}, \infty\right)$, $f(x)$ is a convex function. For $x \in \left[-1, \frac{1}{6}\right]$, $f(x)$ is a concave function. By FOC, the global minimum for $x \in \left[\frac{1}{6}, \infty\right)$ is $x^1 = \frac{1+\sqrt{13}}{6}$. For $x \in \left[-1, \frac{1}{6}\right]$, we may check the boundary of $f(x)$. Since $f(-1) = 0 < f(x^1)$, x^1 is indeed the global minimum for $x \in [-1, \infty)$.

3. (a)

   $$\begin{aligned} \max_{p \geq 0} \quad & (p-10)D(p) \\ \text{s.t.} \quad & D(p) = \begin{cases} 120 - 2p & \text{if } p \in [0, 30] \\ 90 - p & \text{if } p \in [30, 90] \\ 0 & \text{if } p \in (90, \infty) \end{cases} \end{aligned}$$

   (b) In this region, we want to max $f(x) = (p-10)(120-2p)$.
   By FOC, the optimal price is 35. However, 35 is not in the region below 30. Therefore, we may check the boundary and $f(30) = 1200 \geq -1200 = f(0)$. Thus, the optimal price below 30 is $p^1 = 30$.

   (c) In this region, we want to max $f(x) = (p-10)(90-p)$.
   By FOC, the optimal price is $p^2 = 50$. Since p^2 is feasible, it is optimal.

   (d) Combining (b) and (c), we may find the optimal price by comparing p^1 and p^2. Since $f(p^1) = 1200 \leq 1600 = f(p^2)$, $p^2 = 50$ is the optimal price.