Operations Research, Spring 2017
Homework 2

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1 Problems

1. (25 points; 5 points each) A person uses four kinds of materials to make three kinds of products. For each of materials 1, 2, 3, and 4, the daily supply amount is 300, 400, 500, and 600 units. For each of products 1, 2, and 3, the retail price is \$10, \$15, and \$25. To make one product 1, we need 2 units of material 1, 3 units of material 2, 4 units of materials 3, and no material 4. To make one product 2, we need 3 units of material 1, 6 units of material 2, 3 units of materials 3, and 1 unit of material 4. To make one product 3, we need 5 units of material 1, 7 units of material 2, 9 units of materials 3, and 10 unit of material 4. The maximum numbers of product that can be sold are 100, 80, and 50 units for products 1, 2, and 3, respectively. The person wants to find a production and sales plan to maximize its total revenue.

   (a) Formulate a linear program that solves the person's problem.

   (b) Formulate the dual program of the primal program in Part (a).

   (c) You are given a primal optimal solution: producing 10, $\frac{10}{3}$, and 50 units of products 1, 2, and 3, respectively. Without solving the dual program you formulate in Part (b), determine which dual variables will be 0 and which dual constraints will be binding. Briefly explain why.

   (d) Use your argument in Part (c) to solve a linear system whose unique solution gives you an optimal solution to your dual program in Part (b). Verify that your primal and dual optimal solutions satisfies strong duality.

   (e) Suppose that someone approaches you and proposes to sell you some material 2 at \$1.5 per unit. Should you consider it or not? What if it is material 3 for the same price?

2. (20 points) Consider the following integer program:

$$\begin{array}{rl} \min & 3x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 10 \\ & 3x_1 + 4x_2 \geq 25 \\ & x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2 \end{array}$$

   The set $\mathbb{Z}$ is the set of all integers and the set $\mathbb{Z}_+$ is the set of all nonnegative integers. The notation "$x_i \in \mathbb{Z}_+$" therefore means "x_i is an integer and $x_i \geq 0$".

   (a) (10 points) Use branch-and-bound to solve the integer program. Write down the complete branch-and-bound tree and an optimal solution.

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(b) (5 points) For the linear relaxation of the IP, find its dual program.

(c) (5 points) For the linear relaxation of the IP, find the shadow prices of the two constraints. If the way you get your answer requires solving a linear program, show the complete process of solving that linear program.

3. (10 points; 5 points each) When one applies the general branch-and-bound approach to solve an integer program, from time to time she/he may need to choose one out of multiple nodes to branch. Below are two proposed ways of choosing a node to branch. Use your own words to provide some justifications for each of the two ways (i.e., point out some advantages for each of the two ways).

   (a) Branch the node with the highest objective value (for a maximization problem).

   (b) Once a node is branched, all its descendants are branched before any nondescendant.

4. (15 points; 5 points each) Ikuta is traveling from Japan to Taiwan. She bought two bags, each can carry up to 20 kg of items. There are several items that she considers to carry. The weights and degrees of importance of these items are given in the table below. Ikuta wants to maximize the total degree of importance of the items she carry while satisfying the capacity constraint, i.e., each bag cannot carry more than 20 kg.

Item	1	2	3	4	5	6	7	8	9	10	11	12
Importance	7	4	1	3	1	2	5	4	3	5	9	8
Weight (kg)	8	6	3	4	2	7	7	4	2	3	6	5

   Do each of the following problems independently.

   (a) Formulate an integer program that solves her problem.

   (b) Suppose that items 2 and 3 cannot be put in the same bag, items 4, 5, and 6 cannot be put in the same bag, at least two of items 8 to 12 must be carried, and at least one of items 1 and 2 must be carried if item 3 is not carried. Formulate an integer program that solves her problem.

   (c) Suppose that Ikuta wants to balance the weights of the two bags. She therefore use a two-stage method to solve her problem. First, she looks for a plan that maximizes the total degree of importance. Let the maximized total degree of importance be z^*. She then looks for a plan that minimizes the weight difference of the two bags subject to a constraint that the total degree of importance must be exactly z^*.[1] In other words, If only plan can achieve z^*, the stage-2 optimization will result in exactly that plan. Nevertheless, if more than one plan can achieve z^*, the stage-2 optimization will help Ikuta find one with the minimum weight difference *among* those plans achieving z^*. Formulate an integer program that solves the stage-2 problem.

5. (10 points) A city is divided into eight districts. The time (in minutes) it takes an ambulance to travel from one district to another is shown in the table below. The population of each district (in thousands) is as follows: district 1, 40; district 2, 30; district 3, 35; district 4, 20; district 5, 15; district 6, 50; district 7, 45; district 8, 60.

District	1	2	3	4	5	6	7	8
1	0	3	4	6	8	9	8	10
2	3	0	5	4	8	6	12	9
3	4	5	0	2	2	3	5	7
4	6	4	2	0	3	2	5	4
5	8	8	2	3	0	2	2	4
6	9	6	3	2	2	0	3	2
7	8	12	5	5	2	3	0	2
8	10	9	7	4	4	2	2	0

[1] What will happen if Ikuta tries to minimize the weight difference without requiring the total degree of importance to be z^*? The problem can then be trivially solved: Do not put anything in any of the two bags!

The city has two ambulances and wants to locate them to two of the districts. For each district, the *population-weighted firefighting time* is defined as the product of the district population times the amount of time it takes for the closest ambulance to travel to it. Formulate an integer program that can minimize the maximum population-weighted firefighting time among the eight districts.

6. (20 points) Write AMPL programs to solve the following problems. For each problem, copy and paste all your AMPL programs to your answer sheet. Then write down an optimal solution found by AMPL (or conclude that the problem is infeasible or unbounded). Your model file cannot contain any value of any parameter.

   (a) Problem 4b.

   (b) Problem 5.

   Report your optimal solution as a suggestion to the decision maker. Good format and representation are needed. In short, write a good report that can be understood by one who has not taken a course in Operations Research!

2 Submission rules

The deadline of this homework is 2 pm, April 24, 2017. Please put a hard copy of the work into the instructor's mailbox on the first floor of the Management Building 2 by the due time. Works submitted between 2 pm and 3 pm will get 10 points deducted as a penalty. Submissions later than 3 pm will not be accepted. Each student must submit her/his individual work.