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Suggested Solution For Homework 3

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1. (a) $\nabla\pi(p) = ae^{-bp}[b(c-p)+1]$, $\quad \nabla^2\pi(p) = abe^{-bp}[b(p-c)-2]$

   (b) Both of the answers are when $abe^{-bp}[b(p-c)-2] \leq 0 \Rightarrow p \leq \frac{bc+2}{b}$.

   (c) $ae^{-bp}[b(c-p)+1] = 0 \Rightarrow \bar{p} = \frac{bc+1}{b}$
   $\nabla^2\pi(\bar{p}) = -abe^{-bp} \leq 0$
   yes, we can conclude the $\bar{p}$ is global optimal because it's a concave function.

   (d) Because $\nabla\pi(c) = ae^{-bp} > 0$, $\nabla\pi(\infty) = -\infty < 0$ and it's concave function, we can show that $\nabla\pi(p)$ is first positive and then negative. we can conclude the $\bar{p}$ is global optimal because it's a concave function .

2. (a) Let the decision variables be

$$q_1 = \text{supply quantities for product 1},$$
$$q_2 = \text{supply quantities for product 2},$$

$$\begin{aligned} \max_{q_1,q_2} \ \pi(q_1,q_2) =& (a_1 - b_1 q_1)q_1 + (a2 - b_2 q_2)q_2 \\ \text{s.t.} \quad & q_1 + q_2 \leq K \\ & q_1 \geq 0 \\ & q_2 \geq 0 \end{aligned}$$

   We can transform the objective function:

$$\max_{q_1,q_2} = (a_1 - b_1 q_1)q_1 + (a2 - b_2 q_2)q_2 \Rightarrow \min_{q_1,q_2} -\pi(q_1,q_2) = -\Big[(a_1 - b_1 q_1)q_1 + (a2 - b_2 q_2)q_2\Big]$$

   $\nabla^2 - \pi(q_1,q_2) = \begin{bmatrix} 2b_1 & 0 \\ 0 & 2b_2 \end{bmatrix} \geq 0$ it's a convex function. So $\pi(q_1,q_2)$ is a concave function. And the constraint is a linear function. As the result, it is a convex program.

   (b) The Lagrangian is $\mathcal{L}(q|\lambda) = (a_1 - b_1 q_1)q_1 + (a_2 - b_2 q_2)q_2 + \lambda(K - q_1 - q_2)$
   $\frac{\partial \mathcal{L}(q|\lambda)}{\partial q_1} = a_1 - 2b_1 q_1 - \lambda$, $\frac{\partial \mathcal{L}(q|\lambda)}{\partial q_2} = a_2 - 2b_2 q_2 - \lambda$
   The KKT condition for the problem is as follow$\lambda \geq 0$
   i. Primal feasibility: $q_1 + q_2 \leq K$
   ii. Dual feasibility: $a_i - 2b_i q_i - \lambda \quad \forall i = 1, 2$
   iii. Complementary slackness: $\lambda(K - q_1 - q_2) = 0$

   (c) By part(b), $q_i = \frac{a_i - \lambda}{2b_i} \quad \forall i = 1, 2$. Because the constraint may be binding or nonbinding, there are two situation:
   i. If the constraint is binding, then $q_1 + q_2 = K \Rightarrow q_1 = \frac{a_1 - a_2 + 2b_2 K}{2(b_1 + b_2)}$, $q_2 = \frac{a_2 - a_1 + 2b_1 K}{2(b_1 + b_2)}$
   ii. If the constraint is nonbinding, then $q_1 = \frac{a_1}{2b_1}$, $q_1 = \frac{a_2}{2b_2}$
   Combine the above result, we can find an optimal analytical solution:
   i. $\pi(q_1^*, q_2^*) = (a_1 - b_1 \frac{a_1 - a_2 + 2b_2 K}{2(b_1 + b_2)})\frac{a_1 - a_2 + 2b_2 K}{2(b_1 + b_2)} + (a_2 - b_2 \frac{a_2 - a_1 + 2b_1 K}{2(b_1 + b_2)})\frac{a_2 - a_1 + 2b_1 K}{2(b_1 + b_2)}$ when the constraint is binding.
   ii. $\pi(q_1^*, q_2^*) = \frac{a_1^2 b_2 + a_1^2 b_1}{4b_1 b_2}$ when the constraint is nonbinding.

(d) If the constraint is nonbinding, K has no impact on the quantities, because there is no K in the quantities. However, If the constraint is binding, quantities will increase in K, because the quantity is proportional to K. The intuitive explanation: the demand limit K release, we can supply more products.

3. (a) The gradient of f is

$$\begin{bmatrix} 2e^{x_1} \\ x_2 \end{bmatrix}$$

The Hessian of f is

$$\begin{bmatrix} 2e^{x_1} & 0 \\ 0 & 1 \end{bmatrix}$$

(b) The next solution with Newtons method.

$$x^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{1}{2e^2} \begin{bmatrix} 1 & 0 \\ 0 & 2e^2 \end{bmatrix} \begin{bmatrix} 2e^2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) The next solution with gradient descent method.

$$x^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2e^2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(1-e^2) \\ 0 \end{bmatrix}$$

(d) The process of choosing the step size to reaches the lowest point.

$$a_0 = \text{argmin}_{a \geq 0} f(x^0 - a\nabla f(x^0))$$

Where

$$f(2-2ae^2, 2-2a) = 2e^{2(1-ae^2)} + \frac{(2-2a)^2}{2} = g(a)$$

By FOC, $g'(a) = -4\mathrm{e}^{4-2\mathrm{e}^2 a} - 2\,(2-2a) = 0$ when $a \approx 1$

Therefore, we can obtain next solution by the gradient descent method:

$$x^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2e^2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(1-e^2) \\ 0 \end{bmatrix}$$

4. (omitted)

5. Full English names of the instructor : Ling-Chieh Kung.
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