MBA 8023: Optimization
The Simplex Method

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December 11, 2013

Introduction

- Last time we have shown that "if there is an optimal solution, there is an extreme point optimal solution."
- Formally, we have the following:

Proposition 1 (Optimality of extreme points)

Let P be a nonempty polyhedron with at least one extreme point. If $\min\{c^T x | x \in P\}$ has an optimal solution, then it has an optimal solution that is an extreme point of P.

- So we only need to focus on extreme points.
- How to list all extreme points?
- How to (let a computer) verify that a point is an extreme point?
- A **geometric** optimality condition is not enough; we need an **algebraic** optimality condition.
  - Based on that, we may construct our algorithm: **the simplex method**.

Road map

- **Algebraic optimality condition**.
- The simplex method.
- More about the simplex method.

Canonical and standard form LPs

- An LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

is in the **canonical form**.

- An LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

is in the **standard form**.

- They are equivalent:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min \quad & c^T x^+ - c^T x^- \\ \text{s.t.} \quad & Ax^+ - Ax^- + Is = b \\ & x^+,\ x^-,\ s \ge 0,\ s \in \mathbb{R}^m \end{aligned}$$

and

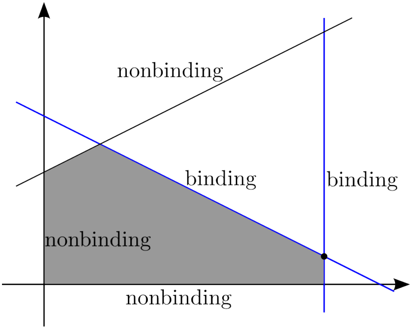
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}. \end{aligned}$$

Binding constraints

- Consider an LP $\min\{c^T x | x \in P\}$ with $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ for some $m \times n$ matrix A. We will assume that $m \geq n$.

Definition 1 (Binding constraint)

Given $\bar{x} \in \mathbb{R}^n$ and a constraint $a^T x \leq b$, we say the constraint is binding or active at $\bar{x}$ if $a^T \bar{x} = b$.



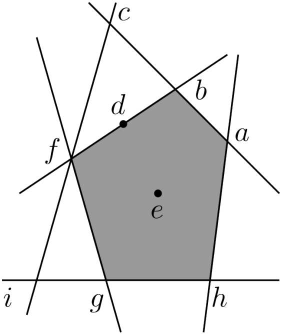
Basic solutions

Definition 2 (Basic solution)

$\bar{x} \in \mathbb{R}^n$ is a basic solution of P if there exist n linearly independent constraints that are binding at $\bar{x}$.

Definition 3 (Basic feasible solution)

$\bar{x} \in \mathbb{R}^n$ is a basic feasible solution of P if it is basic and feasible.



Optimality of basic feasible solutions

Proposition 2 (Optimality of basic feasible solutions)

Let $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. $\bar{x} \in P$ is a basic feasible solution of P if and only if $\bar{x}$ is an extreme point of P.

Proof. ($\Rightarrow$) Suppose $\bar{x}$ is a bfs of P, then there exist n linearly independent binding constraints. Let's partition A into $\begin{bmatrix} A^= \\ A^< \end{bmatrix}$ such that $\begin{bmatrix} A^= \\ A^< \end{bmatrix} \bar{x} = \begin{bmatrix} b^= \\ b^< \end{bmatrix} < b$, then $A^=$ has at least n rows. In addition, we know that there exists an $n \times n$ nonsingular $\tilde{A}$ which is a submatrix of $A^=$. Suppose there exist $x^1, x^2 \in P$ such that $x^1 \neq x^2$ and $\bar{x} = \lambda x^1 + (1 - \lambda) x^2$ for some $\lambda \in (0, 1)$, then

$$\tilde{b} = \tilde{A}\bar{x} = \lambda \tilde{A} x^1 + (1 - \lambda) \tilde{A} x^2 \leq \lambda \tilde{b} + (1 - \lambda) \tilde{b} = \tilde{b},$$

so $\tilde{b} = \tilde{A} x^1 = \tilde{A} x^2$. Then the nonsingularity of $\tilde{A}$ implies that $x^1 = x^2$, which is a contradiction.

Optimality of basic feasible solutions

Proof continued. ($\Leftarrow$) Recall that we partition A into $\begin{bmatrix} A^= \\ A^< \end{bmatrix}$ such that

$$\begin{bmatrix} A^= \\ A^< \end{bmatrix} \bar{x} = \begin{bmatrix} b^= \\ b^< \end{bmatrix} < b.$$

Suppose $\bar{x}$ is not a bfs, then rank $A^= < n$, i.e., $\dim \mathcal{N}(A^=) > 0$. Let $0 \neq y \in \mathcal{N}(A^=)$, i.e., $y \neq 0, A^= y = 0$; also let $x^1 = \bar{x} + \epsilon y$, $x^2 = \bar{x} - \epsilon y$ for some $\epsilon > 0$. Then

$$A^= x^1 = A^=(\bar{x} + \epsilon y) = A^= \bar{x} + \epsilon A^= y = A^= \bar{x}$$

and

$$A^< x^1 = A^<(\bar{x} + \epsilon y) = A^< \bar{x} + \epsilon A^< y = b^< + \epsilon A^< y < b$$

for ϵ sufficiently small. So $x^1 \in P$. Similarly, $x^2 \in P$, and thus $\bar{x} = \frac{1}{2}x^1 + \frac{1}{2}x^2$. As $y \neq 0$, we know $x^1 \neq x^2$. Therefore, $\bar{x}$ is not an extreme point. □

Enumerating basic feasible solutions

- Now we only need to list all basic feasible solutions.
- Checking whether a point is a basic feasible solution is easy.
- Enumerating all of them can also be done **systematically**.
  - Pick n constraints out of the m ones.
  - Check whether they are linearly independent (how?).
  - Set them to binding and find a basic solution (how?).
  - Check whether it is feasible.
- However, this is impractical!
  - There are $\binom{m}{n}$ distinct ways of selecting constraints. Still **too many**!
  - It is uneasy to deal with **infeasible** and **unbounded** LPs.
- We need a "clever way" to search among basic feasible solutions.
  - The simplex method is the clever way.
  - It is for standard form LPs.

Basic feasible solutions for standard form LPs

- Consider a standard form LP

$$
(P) \quad \begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax = b & (m \text{ equalities}) \\ & x \geq 0 & (n \text{ inequalities}). \end{array}
$$

Definition 4 (Basic solutions for standard form LPs)

$\bar{x} \in \mathbb{R}^n$ is a basic solution of (P) if there exists a partition of A into $[A_B \ A_N]$ and of $\bar{x}$ into $(\bar{x}_B, \bar{x}_N)$ such that A_B is a nonsingular $m \times m$ matrix, $\bar{x}_B = A_B^{-1} b$, and $\bar{x}_N = 0$.

  - Among the n inequalities, select $n - m$ of them to be binding.
- Among the n variables, select m of them to be **basic**:
  - Variables x_is, $i \in B$, are **basic variables**.
  - Variables x_js, $j \in N$, are **nonbasic variables**. $x_j = 0$ for all $j \in N$.
  - B is called the **basis** of the basic solution.
- Note that $\bar{x} = (\bar{x}_B, \bar{x}_N)$ is a basic feasible solution if $\bar{x}_B = A_B^{-1} b \geq 0$.

Basic feasible solutions for standard form LPs

- As an example, consider a standard form LP with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

- There are three ways of selecting $m = 2$ basic variables out of the $n = 3$ variables:
  - Let $B = \{1, 2\}$, $N = \{3\}$, then $A_B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $A_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_B^{-1} b = (x_1, x_2) = (2, 1)$, $x_3 = 0$. We then have $\bar{x} = (2, 1, 0)$ as a basic feasible solution.
  - Let $B = \{2, 3\}, N = \{1\}$, then $\bar{x} = (0, 3, 2)$ is a basic feasible solution.
  - Let $B = \{1, 3\}, N = \{2\}$, then $\bar{x} = (3, 0, -1) \not\geq 0$ is not a basic feasible solution.
- The **order** matters!

Road map

- Algebraic optimality condition.
- **The simplex method**.
- More about the simplex method.

The simplex method

- We now consider solving a standard form LP

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array}$$

- We may assumed that rank $A = m$ WLOG.
  - Otherwise, we can just remove those redundant constraints.
- **The simplex method** proceeds as follows: Given a basic feasible solution $x = (x_B, x_N)$ in each iteration, try to move to another **strictly better** basic feasible solution (i.e., one with a strictly lower objective value).
  - Greedy search: A local minimum is a global minimum.
  - Search among extreme points only.
- How to do it **algebraically**?

The reduced form

- First, we rewrite (P) as

$$\begin{aligned} \min \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0. \end{aligned}$$

Because

$$\begin{aligned} & A_B x_B + A_N x_N = b \\ \Leftrightarrow\ & x_B = A_B^{-1}(b - A_N x_N) = A_B^{-1} b - A_B^{-1} A_N x_N, \end{aligned}$$

(P) can be further reduced to (P'):

$$\begin{aligned} \min \quad & c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\ \text{s.t.} \quad & A_B^{-1} b - A_B^{-1} A_N x_N \geq 0,\ x_N \geq 0. \end{aligned}$$

- $\bar{c}_N = c_N^T - c_B^T A_B^{-1} A_N$ is the **reduced costs** of the nonbasic set N.
- Recall that $x_N = 0$. Therefore, $c_B^T A_B^{-1} b$ is the objective value of B.

Making an improvement

$$\begin{aligned} \min \quad & c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\ \text{s.t.} \quad & A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, \ x_N \geq 0 \end{aligned}$$

- Looking at the objective function. If there exists $j \in N$ such that the reduced cost
$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j < 0,$$
we can **increase** x_j (which is a nonbasic variable and is 0 currently) to lower the objective value.
- We should keep increasing x_j as long as we **satisfy the constraints**.
  - Obviously, $x'_N \geq 0$ will still be satisfied.
  - How to check $x'_B = A_B^{-1} b - A_B^{-1} A_N x'_N = A_B^{-1} b - A_B^{-1} A_j x'_j \geq 0$?

When to stop?

$$\begin{array}{ll} \min & c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\ \text{s.t.} & A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, \ x_N \geq 0 \end{array}$$

- Let $\bar{b} = A_B^{-1} b \geq 0$ and $d = A_B^{-1} A_j$, then

$$x'_B = \bar{b} - x'_j d = \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} - x'_j \begin{bmatrix} + \\ - \\ 0 \\ + \end{bmatrix} \geq 0$$

$$\Leftrightarrow \alpha^* = \min_{i \in B} \left\{ \frac{\bar{b}_i}{d_i} \middle| d_i > 0 \right\} \text{ and } x'_j \in [0, \alpha^*].$$

- We will increase x_j to $x'_j = \alpha^*$.
- This will make x_l becomes $x'_l = 0$, where

$$l \in \operatorname*{argmin}_{i \in B} \left\{ \frac{\bar{b}_i}{d_i} \middle| d_i > 0 \right\}.$$

Entering and leaving variables

$$\begin{aligned} \min \quad & c_B^T A_B^{-1} b + (c_N^T - c_B^T A_B^{-1} A_N) x_N \\ \text{s.t.} \quad & A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, \; x_N \geq 0 \end{aligned}$$

- We have chosen to increase x_j, where its reduced cost

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j < 0.$$

- We stop when $x_j = \alpha^*$, where

$$l \in \operatorname*{argmin}_{i \in B} \left\{ \frac{\bar{b}_i}{d_i} \middle| d_i > 0 \right\} \quad \text{and} \quad \alpha^* = \frac{\bar{b}_l}{d_l}.$$

- Originally, $x_j = 0$ and $x_l > 0$. Now $x_j > 0$ and $x_l = 0$.[1]
- We say that x_j **enters** the basis and x_l **leaves** the basis.
  - x_j is the **entering variable**.
  - x_l is the **leaving variable**.

[1] If $\bar{b}_l = 0$, $x_j = 0$. we will ignore such a degenerate case in this lecture.

The algorithm

- The simplex method can now be summarized below:

  (Initialization) Input a basic feasible solution (x_B, x_N), where $x_B = A_B^{-1} b \geq 0$ and $x_N = 0$.

  1. (Entering) Let $\bar{c}_N = c_N - c_B^T A_B^{-1} A_N$.
     1.1 If for all $j \in N$ we have $\bar{c}_j \geq 0$, (x_B, x_N) is optimal and we stop.[2]
     1.2 Otherwise, pick an x_j with $\bar{c}_j < 0$.
  2. (Leaving) Let $d = A_B^{-1} A_j$ and $\alpha^* = \min_{i \in B}\{\frac{\bar{b}_i}{d_i} | d_i > 0\}$ where $\bar{b} = A_B^{-1} b$.
     2.1 If for all $i \in B$ we have $d_i \leq 0$, the problem is unbounded and we stop.
     2.2 Otherwise, let $l \in \text{argmin}_{i \in B}\{\frac{\bar{b}_i}{d_i} | d_i > 0\}$, set $x_l = 0$, set $x_j = \alpha^*$, replace B by $B \cup \{j\} \setminus \{l\}$, and replace N by $N \cup \{l\} \setminus \{j\}$. Go to 1 and repeat.
- Remaining questions:
  - How to find an initial basic feasible solution?
  - Is A_B always invertible?
  - How to select an entering/leaving variable among multiple candidates?

[2]Because a local minimum is a global minimum.

An example

- Consider the LP

$$\begin{array}{rrcrcrcrcl} \min & 2x_1 & & & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & 2x_1 & & & + & 3x_3 & + & 4x_4 & = & 2 \\ & \multicolumn{9}{l}{x_1,\ x_2,\ x_3,\ x_4 \geq 0.} \end{array}$$

- (Initialization) If $B = \{1, 2\}$ and $N = \{3, 4\}$, we have

$$A_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, A_N = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, x_B = A_B^{-1} b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  So $x^0 = (1, 1, 0, 0)$ can be an initial basic feasible solution.
- (Iteration 1) Compute $\bar{c}_N^T = c_N^T - c_B^T A_B^{-1} A_N$ as

$$\begin{bmatrix} \bar{c}_3 & \bar{c}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \end{bmatrix} < 0.$$

  Let's enter x_3.

An example

- (Iteration 1 continued) Now we have

$$x'_B = A_B^{-1}b - A_B^{-1}A_3x'_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} x'_3.$$

Since only $\frac{3}{2} > 0$, we let $x'_3 = \frac{1}{\frac{3}{2}} = \frac{2}{3}$ and $x'_1 = 0$. The current solution $x^1 = (0, \frac{4}{3}, \frac{2}{3}, 0)$ is better (why $x_2 = \frac{4}{3}$?)

- (Iteration 2) Now, $B = \{3, 2\}$, $N = \{1, 4\}$, and[3]

$$\begin{aligned} \bar{c}_N^T &= \begin{bmatrix} \bar{c}_1 & \bar{c}_4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix} \geq 0. \end{aligned}$$

Therefore, the current solution $x^1 = (0, \frac{4}{3}, \frac{2}{3}, 0)$ is optimal.

[3] Keep an eye on how the columns of A_B and A_N are ordered. Those orders must be consistent with those of c_B and c_N!

Road map

- Algebraic optimality condition.
- The simplex method.
- **More about the simplex method**.
  - Finding an initial basic feasible solution.
  - The invertiability of A_B.
  - The rule for selecting entering/leaving variable.

Initial basic feasible solution

- To find an initial basic feasible solution (or show that there is none), we may apply the **two-phase method**.
- Given (P), we construct a phase-I LP (Q):[4]

$$(P) \quad \begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (Q) \quad \begin{array}{rl} \min & 1^T y \\ \text{s.t.} & Ax + Iy = b \\ & x, y \geq 0. \end{array}$$

- (Q) has a basic feasible solution $(x, y) = (0, b)$, so we can apply the simplex method on (Q).
- Key: (P) is **feasible** if and only if (Q) has an optimal objective value 0.
- After we solve (Q), either we know (P) is infeasible or the optimal solution for (Q), $(\bar{x}, \bar{y}) = (\bar{x}, 0)$, gives up a basic feasible solution for (P), $\bar{x}$.
- Then we can apply the simplex method to (P).

[4] Even if in (P) we have a maximization objective function, (Q) is still the same.

Example

- To find an initial basic feasible solution (or show that there is none), we may apply the **two-phase method**.
- Given (P), we construct a phase-I LP (Q):

$$
(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (Q) \quad \begin{array}{ll} \min & 1^T y \\ \text{s.t.} & Ax + Iy = b \\ & x, y \geq 0. \end{array}
$$

- (Q) has a basic feasible solution $(x, y) = (0, b)$, so we can apply the simplex method on (Q).
- (P) is feasible if and only if (Q) has an optimal value 0.
- After we solve (Q), either we know (P) is infeasible or the optimal solution for (Q), $(\bar{x}, \bar{y}) = (\bar{x}, 0)$, gives up a basic feasible solution for (P), $\bar{x}$.
- Then we can apply the simplex method to (P).

Invertiability of the basic matrix

- At each iteration, we replace the column A_l in A_B by A_j to get A'_B.
- Is such A'_B still **nonsingular**?
  - With A_j, we do $d = A_B^{-1} A_j$ and $l = \text{argmin}_i \{ \frac{\bar{b}_i}{d_i} : d_i > 0 \}$ to get A_l.
  - $d = A_B^{-1} A_j \Leftrightarrow A_j = A_B d$.
  - So we can write

$$A'_B = \begin{bmatrix} | & | & | & | & | & | & | \\ A_1 & \cdots & A_{l-1} & A_j & A_{l+1} & \cdots & A_m \\ | & | & | & | & | & | & | \end{bmatrix} = A_B I_{ld},$$

  where

$$I_{ld} = \begin{bmatrix} | & | & | & | & | & | & | \\ e_1 & \cdots & e_{l-1} & d & e_{l+1} & \cdots & e_m \\ | & | & | & | & | & | & | \end{bmatrix}.$$

  - $\det A'_B = \det A_B \det I_{ld}$, so $\det A'_B \neq 0$ if and only if $\det I_{ld} \neq 0$.
  - By $d_l > 0$ (why?), we know $\det I_{ld} \neq 0$, so A'_B is nonsingular.

Degeneracy

- Why is variable selection rule important?
- In general, an LP may be **degenerate**.

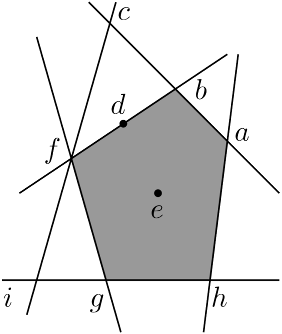
Definition 5

A basic solution $\bar{x}$ is degenerate if there are more than n binding constraints of $\bar{x}$.

Definition 6

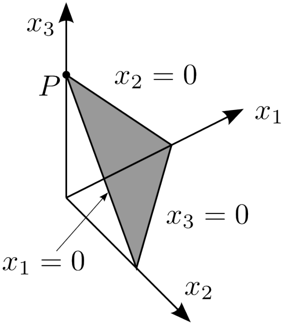
An LP is degenerate if there is at least one degenerate basic feasible solution.

- What may happen when we run the simplex method to a degenerate LP?



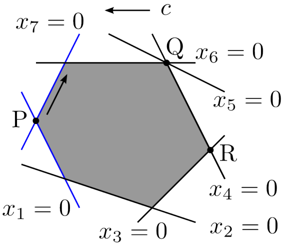
Feasible region of standard form LPs

- Let's become more familiar with constraints in a standard-form LP first.
- For a standard form LP with A being 1×3, there are three variables and one constraint.
  - Each side of this triangle can be expressed by **a nonnegativity constraint** $x_i = 0$.
- At P, the nonbasic set is $N = \{1, 2\}$.
  - At each basic feasible solution, $j \in N$ means that $x_j \geq 0$ is **binding**.
- When we run the simplex method on standard form LPs, we move along **edges**.
  - We move along **binding nonnegativity constraints**.



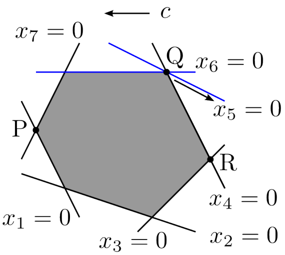
No improvement in an iteration

- In this example, A is 5×7.
- The optimal solution is point R.
- The initial basic feasible solution is point P.
  - At point P, the two binding constraints are $x_1 \geq 0$ and $x_7 \geq 0$.
  - Moving along either one is improving.
  - Suppose we move along $x_7 \geq 0$.
- We stop when we hit $x_6 \geq 0$.
  - x_1 enters and x_6 leaves.
  - The set of binding constraints becomes $x_6 \geq 0$ and $x_7 \geq 0$.
  - Only moving along $x_6 \geq 0$ is improving.
- We stop when we hit... what?



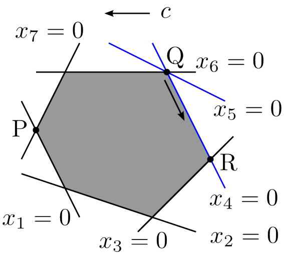
No improvement in an iteration

- If we move along $x_6 \geq 0$, we arrive point Q.
- We hit **two** constraints **at the same time**.
  - We hit both $x_4 \geq 0$ and $x_5 \geq 0$.
  - In simplex, we will choose one of them into the set of binding constraints.
- If we (unluckily) choose to include $x_5 \geq 0$:
  - x_7 enters and x_5 leaves.
  - At this moment, $x_4 = 0$ is treated as basic.
- We now may move along $x_6 \geq 0$ or $x_5 \geq 0$.
  - Moving along $x_6 \geq 0$ is not improving.
  - Moving along $x_5 \geq 0$ is improving.
- However, we hit $x_4 \geq 0$ **immediately**!
  - In this iteration, we move "from Q to Q".
  - It is possible to have **no improvement** in a simplex iteration.



No improvement in an iteration

- We hit $x_4 \geq 0$ when we move along $x_5 \geq 0$.
  - So the set of binding constraints becomes $x_5 \geq 0$ and $x_4 \geq 0$.
  - x_6 enters and x_4 leaves.
- We may now move along $x_4 \geq 0$ and move to the optimal point R.
- In general, we may get stock at a basic feasible solution **forever**!
  - When we do not apply a "good" variable selection rule.



Variable selection rule

- To guarantee that the simplex terminates, we need a well-designed variable selection rule.

Proposition 3 (The smallest index rule)

Using the following rule guarantees to solve an LP in finite steps:

- *Among nonbasic variables with* $\bar{c}_j < 0$*, pick the one with the smallest index to enter the basis.*
- *Among basic variables that minimizes* $\frac{\bar{b}_i}{d_i}$*, pick the one with smallest index to exist.*