MBA 8023: Optimization
Game Theory

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Introduction

- So far we have focused on decision making problems with only one decision maker.
- **Game theory** provides a rigorous framework for analyzing **multi-player** decision making problems.
- As we will see, Linear Programming and Nonlinear Programming are foundations for analyzing games.
  - Dynamic Programming is a foundation for analyzing dynamic games.

Road map

- **Introduction**.
- Nash equilibrium.
- Mixed strategies.
- Zero-sum games.
- Zero-sum games and duality.

Prisoners' dilemma: story

- A and B broke into a grocery store and stole some money. Before police officers caught them, they hided those money carefully without leaving any evidence. However, a monitor got their images when they broke the window.
- They were kept in two separated rooms. Each of them were offered two choices: **Denial or confession**.
  - If both of them deny the fact of stealing money, they will both get one month in prison.
  - If one of them confesses while the other one denies, the former will be set free while the latter will get nine months in prison.
  - If both confesses, they will both get six months in prison.
- They **cannot communicate** and they must make their choices **simultaneously**.
- What will they do?

Prisoners' dilemma: matrix representation

- We may use the following matrix to summarize this "game":

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

  - There are two **players**, player 1 chooses actions in rows and player 2 chooses actions in columns.
  - For each combination of actions, the two numbers are the **payoffs** of the two players under their actions: the first for player 1 and the second for player 2.
  - E.g., if both prisoners deny, they will both get one month in prison, which is represented by a payoff of -1.
  - E.g., if prisoner 1 denies and prisoner 2 confesses, prisoner 1 will get 0 month in prison (and thus a payoff 0) and prisoner 2 will get 9 months in prison (and thus a payoff -9).

Prisoners' dilemma: solution

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Prisoner 1 thinks:
  - "If he denies, I should confess."
  - "If he confesses, I should still confess."
  - "I see! I should confess anyway!"
- For prisoner 2, the situation is the same and he will also **confess**.
- The **solution** of this game, i.e., the **outcome**, is that both prisoner will confess.
  - This is people's **prediction** of this game.
- This outcome can be "improved" if they can **cooperate**.

Prisoners' dilemma: discussions

- A game like the prisoners' dilemma in which all players choose their actions **simultaneously** is called a **static game**.
- This question (with a different story) was first formally raised by Professor Tucker (one of the names in the KKT condition) in a seminar.
- In this game, confession is said to be a **dominant strategy**.
- It illustrates that **lack of coordination** can result in a **lose-lose** outcome.
  - This situation is termed as **socially inefficient**.
- Interestingly, even if they promised each other to deny once they are caught, this promise is **non-credible**. Both of them will still confess to maximize their payoffs.

Prisoners' dilemma: Advertising game

- Two companies are competing in a market.
- At this moment, they both earn four million dollars per year.
- Each of them may choose to advertise with a cost of three million per year:
  - If one advertises while the other does not, she earns nine millions and the competitor earns one million.
  - If both advertise, both will earn six millions.

	Advertise	Be silent
Advertise	$3, 3$	$6, 1$
Be silent	$1, 6$	$4, 4$

- What will they do?

Prisoners' dilemma: Arms race

- Two countries are neighbors.
- Each of them may choose to develop a new weapon:
  - If one does so while the other one keep the current status, the former's payoff is 20 and the latter's payoff is -100.
  - If both do this, however, their payoffs are both -10.

	NW	CS
NW	$-10, -10$	$20, -100$
CS	$-100, 20$	$0, 0$

- What will they do?

Predicting the outcome of other games

- How about games that are not the prisoners' dilemma? Do we have a systematic way to predict the outcome?
- What will be the outcome (a combination of actions chosen by the two players) of the following game?

	Left	Middle	Right
Up	$1, 0$	$1, 2$	$0, 1$
Down	$0, 3$	$0, 1$	$2, 0$

Eliminating strictly dominated options

- We may apply the same trick we used to solve the prisoners' dilemma.
- For player 2, playing Middle **dominates** playing Right. So we may **eliminate** the column of Right without eliminating any possible outcome:

	Left	Middle	Right
Up	$1, 0$	$1, 2$	$0, 1$
Down	$0, 3$	$0, 1$	$2, 0$

$\rightarrow$

	Left	Middle
Up	$1, 0$	$1, 2$
Down	$0, 3$	$0, 1$

Eliminating strictly dominated options

- Now, player 1 knows that player 2 will never play Right.
- Facing the reduced game, player 1 finds that playing Down is dominated by playing Up.
- The row of Down can thus be eliminated:

	Left	Middle
Up	$1, 0$	$1, 2$
Down	$0, 3$	$0, 1$

$\rightarrow$

	Left	Middle
Up	$1, 0$	$1, 2$

- Knowing that player 1 will only choose Up, player 2 will simply choose Middle.
- The outcome of this game will be that player 1 chooses Up and player 2 chooses Middle.

Eliminating strictly dominated options

- In game theory, options are typically called **strategies**.
- The above idea is called **iterative elimination** of **strictly dominated strategies**.
- It solves some games. However, is also fails to solve some others.
- Consider the following game "Matching pennies":

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- What may we do when no more strategies can be eliminated?
- In 1950, John Nash formalized the concept of **equilibrium solutions**, which are called **Nash equilibria** nowadays.[1]

[1] He did that as a Ph.D. students, when he was 22 years old.

Road map

- Introduction.
- **Nash equilibrium**.
- Mixed strategies.
- Zero-sum games.
- Zero-sum games and duality.

Nash equilibrium: definition

- The most fundamental equilibrium concept, Nash equilibrium, is defined as follows:

Definition 1

For an n-player game, let S_i be player i's action space and u_i be player i's utility function, $i = 1, ..., n$. An action profile $(s_1^*, ..., s_n^*)$, $s_i^* \in S_i$, is a Nash equilibrium if

$$u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \geq u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

for all $s_i \in S_i$, $i = 1, ..., n$.

- In other words, s_i^* solves

$$\max_{s_i \in S_i} \quad u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*).$$

Nash equilibrium: an example

- Consider the following game in which no strategy/action is strictly dominated:

	L	C	R
T	$0, 4$	$4, 0$	$5, 3$
M	$4, 0$	$0, 4$	$5, 3$
B	$3, 5$	$3, 5$	$6, 6$

- What is a Nash equilibrium?
  - (T, L) is not: Player 1 will deviate to M or B.
  - (T, C) is not: Player 2 will deviate to L or R.
  - (B, R) is: No one will unilaterally deviate.
  - Any other Nash equilibrium?

Nash equilibrium as a solution concept

	L	C	R
T	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
B	3, 5	3, 5	6, 6

- In a static game, a Nash equilibrium is a reasonable outcome.
  - Imagine that the players play this game **repeatedly**.
  - If they happen to be in a Nash equilibrium, no one has the incentive to **unilaterally deviate**, i.e., to change her action while all others keep their actions.
  - If they do not, at least one will deviate. This process will continue until a Nash equilibrium is reached.
- For example, if they starts at (T, L), eventually they will stop at (B, R), the unique Nash equilibrium of this game.

Nash equilibrium: More examples

- Is there any Nash equilibrium of the prisoners' dilemma?

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Is there any Nash equilibrium of the game "BoS"?
  - Battle of sexes.
  - Bach or Stravinsky.

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- Is there any Nash equilibrium of the matching pennies game?

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Cournot Competition

- In 1838, Antoine Cournot introduced the following **quantity competition** between two retailers.
- Let q_i be the production quantity of firm i, $i = 1, 2$.
- Let $P(Q) = a - Q$ be the market-clearing price for an aggregate demand $Q = q_1 + q_2$.
- Unit production cost of both firms is $c < a$.
- Our questions are:
  - In this environment, what will these two firms do?
  - Is the outcome satisfactory?
  - What is the difference between duopoly and monopoly (or equivalently, decentralization or integration).

Cournot Competition

- Players: 1 and 2.
- Action spaces: $S_i = [0, \infty)$ for $i = 1, 2$.
- Utility functions:

$$u_i(q_1, q_2) = q_i[a - (q_i + q_{3-i}) - c], i = 1, 2.$$

- As for an outcome, we look for a Nash equilibrium.
- If (q_1^*, q_2^*) is a Nash equilibrium, it must satisfy

$$q_1^* = \operatorname*{argmax}_{q_1 \in [0,\infty)} u_1(q_1, q_2^*) = \operatorname*{argmax}_{q_1 \in [0,\infty)} q_1[a - (q_1 + q_2^*) - c] \text{ and}$$
$$q_2^* = \operatorname*{argmax}_{q_2 \in [0,\infty)} u_2(q_1^*, q_2) = \operatorname*{argmax}_{q_2 \in [0,\infty)} q_2[a - (q_1^* + q_2) - c].$$

Solving the Cournot competition

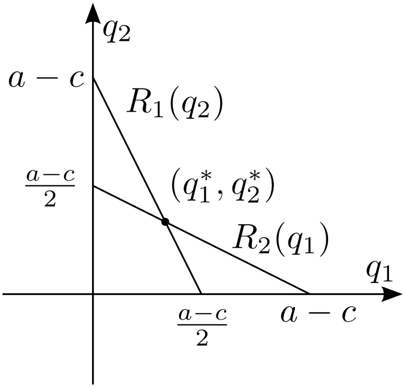
- For firm 1's problem, we first see that it is a convex program:
  - $u_1'(q_1, q_2^*) = a - q_1 - q_2^* - c - q_1$.
  - $u_2''(q_1, q_2^*) = -2 < 0$.
- The FOC condition suggests $q_1^* = \frac{1}{2}(a - q_2^* - c)$. As long as $q_2^* < a - c$, q_1^* is optimal for firm 1.
- Similarly, $q_2^* = \frac{1}{2}(a - q_1^* - c)$ is firm 2's optimal decision as long as $q_1^* < a - c$.
- So if (q_1^*, q_2^*) is a Nash equilibrium, it must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

- The unique solution to this system is $q_1^* = q_2^* = \frac{a-c}{3}$.
  - Does this solution make sense?
  - This is indeed the unique Nash equilibrium as $\frac{a-c}{3} < a - c$.

Best responses

- Another way of solving this game is to use the **best response** functions.
  - Given the other player's any decision, what is my optimal decision?
- Firm 1's best response to firm 2 is $R_1(q_2) = \frac{1}{2}(a - q_2 - c)$.
- Similarly, firm 2's best response is $R_2(q_1) = \frac{1}{2}(a - q_1 - c)$.
- A Nash equilibrium always lies on an **intersection** of the two best response functions.



Distortion due to decentralization

- Suppose the two firms' are **integrated** together to jointly choose the aggregate production quantity.
- They together solve

$$\max_{Q \in [0,\infty)} \quad Q[a - Q - c],$$

  whose optimal solution is $Q^* = \frac{a-c}{2}$.
- Note that $Q^* = \frac{a-c}{2} < \frac{2(a-c)}{3} = q_1^* + q_2^*$.
- Why does a firm intend to **increase** its production quantity under decentralization?

Inefficiency due to decentralization

- May these firms improve their profitability with integration?
- Under decentralization, firm i earns

$$\pi_i^D = \frac{(a-c)}{3}\left[a - \frac{2(a-c)}{3} - c\right] = \left(\frac{a-c}{3}\right)\left(\frac{a-c}{3}\right) = \frac{(a-c)^2}{9}.$$

- Under integration, the two firms earn

$$\pi^C = \frac{(a-c)}{2}\left[a - \frac{a-c}{2} - c\right] = \left(\frac{a-c}{2}\right)\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{4}.$$

- $\pi^C > \pi_1^D + \pi_2^D$: The integrated system is more **efficient**.
- Through appropriate profit splitting, both firm earns more.
  - Integration is a **win-win** solution!

Inefficiency due to decentralization

- How about consumers?
- Under decentralization, the aggregate quantity is $\frac{2(a-c)}{3}$ and the market-clearing price is $\frac{a-c}{3}$.
- Under integration, the aggregate quantity is $\frac{a-c}{2}$ and the market-clearing price is $\frac{a-c}{2}$.
- Under decentralization, **more** consumers buy this product with a **lower** price.
- Consumers **benefits from competition**.
- Integration benefits the firms but hurts consumers.

The two firms' prisoners' dilemma

- Now we know it is the two firms' best interests to together produce $Q = \frac{a-c}{2}$.
- What if we suggest each of them to choose $q_1' = q_2' = \frac{a-c}{4}$?
- This results in $Q = \frac{a-c}{2}$, which maximizes the total profit.
- However, this is **not** a Nash equilibrium:
  - "If the other firm chooses $q' = \frac{a-c}{4}$, I will move to
    $$q'' = R(q') = \frac{1}{2}(a - q' - c) = \frac{3(a-c)}{8}.$$
  - So both firms will have incentives to unilaterally deviate.
- These two firms are engaged in a prisoners' dilemma!

Bertrand competition

- In 1883, Joseph Bertrand considered another format of retailer competition: They choose **prices** instead of quantities.
- Firm i chooses price p_i, $i = 1, 2$.
- Firm i's demand quantity is

$$q_i = a - p_i + bp_{3-i}, i = 1, 2.$$

  - $b \in [0, 1)$ measures the **intensity of competition** is: The larger b, the more intense the competition.
  - Why $b < 1$?
- Unit production cost $c < a$.

Solving the Bertrand competition

- Suppose (p_1^*, p_2^*) is a Nash equilibrium.
- For firm 1, p_1^* must be an optimal solution of

$$\max_{p_1 \in [0,\infty)} \pi_1\big(p_1, p_2^*\big) = \big(a - p_1 + bp_2^*\big)(p_1 - c).$$

  It can be verified that $p_1^* = \frac{1}{2}(a + bp_2^* + c)$.
- Similarly, $p_2^* = \frac{1}{2}(a + bp_1^* + c)$.
- The unique Nash equilibrium is $p_1^* = p_2^* = \frac{a+c}{2-b}$.
  - Does this solution make sense?

Distortion due to decentralization

- Under integration, the two firms together choose a **single price** P to solve

$$\max_{P \in [0,\infty)} \; 2(a - P + bP)(P - c),$$

whose optimal solution P^* satisfies the FOC

$$\begin{aligned} & (-1+b)(P^* - c) + a - P^* + bP^* = 0 \\ \Leftrightarrow\ & (-1+b)P^* + a + c(1-b) = 0 \\ \Leftrightarrow\ & P^* = \frac{a + c(1-b)}{2(1-b)}. \end{aligned}$$

- Is $P^* > p_1^* = p_2^*$?

$$P^* > p_1^* \ \Leftrightarrow \frac{a + c(1-b)}{2(1-b)} > \frac{a+c}{2-b} \Leftrightarrow a > c(1-b).$$

Is $a > c(1-b)$ always true?

Road map

- Introduction.
- Nash equilibrium.
- **Mixed strategies**.
- Zero-sum games.
- Zero-sum games and duality.

Mixed strategy

- Choosing a single action deterministically is said to implement a **pure strategy**.
- A **mixed strategy** for player i is a **probability distribution** over the strategy space S_i.
  - She **randomizes** her choice of actions with the distribution.
  - E.g., in the matching penny game, player 1's mixed strategy is a probability distribution $(q, 1-q)$, where $\Pr(\text{Head}) = q$ and $\Pr(\text{Tail}) = 1-q$.
- Formally, if all the strategy spaces are finite and of size K_i:

Definition 2

A mixed strategy for player i is a vector $p_i = (p_{i1}, ..., p_{iK_i})$, where $0 \leq p_{ij} \leq 1$ for all $j = 1, ..., K_i$ and $\sum_{j=1}^{K_i} p_{ij} = 1$.

Mixed-strategy Nash equilibrium

- A profile is a **mixed-strategy Nash equilibrium** if no player can unilaterally deviate (modify her own mixed strategy) and obtain a strictly higher **expected** utility.
- Let's use the matching penny game as an example.

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

  - Let $(q, 1-q)$ be player 1's mixed strategy.
  - Let $(r, 1-r)$ be player 2's mixed strategy.

Mixed-strategy Nash equilibrium

- Under their strategies, player 1 will earn:
  - $u_1(H, H) = 1$ with probability qr.
  - $u_1(H, T) = -1$ with probability $q(1 - r)$.
  - $u_1(T, H) = -1$ with probability $(1 - q)r$.
  - $u_1(T, T) = 1$ with probability $(1 - q)(1 - r)$.
- Player 1's expected utility is

$$\begin{aligned} & v_1(q, r) = \mathbb{E}[u_1(q, r)] \\ = & \; qru_1(H, H) + q(1 - r)u_1(H, T) \\ & + (1 - q)ru_1(T, H) + (1 - q)(1 - r)u_1(T, T) \\ = & \; qr + (1 - q)(1 - r) - q(1 - r) - (1 - q)r \\ = & \; 4qr - 2q - 2r + 1 = 2q(2r - 1) - 2r + 1. \end{aligned}$$

- Similarly, player 2's expected utility is

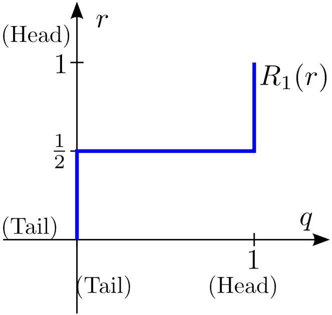
$$v_2(q, r) = -4qr + 2q + 2r - 1 = 2r(-2q + 1) + 2q - 1.$$

Mixed-strategy Nash equilibrium

- For player 1, let $q^* = R_1(r)$ be the best response that maximizes

$$v_1(q, r) = 2q(2r - 1) - 2r + 1.$$

  - If $r < \frac{1}{2}$, $R_1(r) = 0$.
  - If $r > \frac{1}{2}$, $R_1(r) = 1$.
  - If $r = \frac{1}{2}$, $R_1(r) = [0, 1]$ (q does not matter).

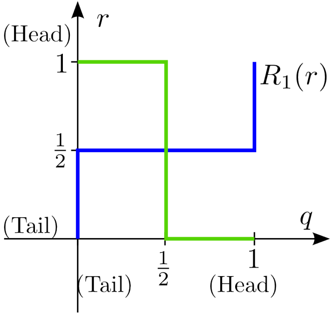


Mixed-strategy Nash equilibrium

- For player 2, the best response that maximizes

$$v_2(q, r) = -4qr + 2q + 2r - 1 = 2r(-2q + 1) + 2q - 1.$$

is $r^* = R_2(q) = 1$ if $q < \frac{1}{2}$, 0 if $q > \frac{1}{2}$, and $[1, 0]$ if $q = \frac{1}{2}$.



- The unique mixed-strategy Nash equilibrium is $(q^*, r^*) = (\frac{1}{2}, \frac{1}{2})$.

BoS

- Consider the game BoS as another example.

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- There are two pure-strategy Nash equilibria. Which two?
  - They are also mixed-strategy Nash equilibria.
  - Is there other mixed-strategy Nash equilibrium?
- Mixed strategies:
  - Let $(q, 1-q)$ be player 1's mixed strategy: $\Pr(B) = q = 1 - \Pr(S)$.
  - Let $(r, 1-r)$ be player 2's mixed strategy: $\Pr(B) = r = 1 - \Pr(S)$.

BoS

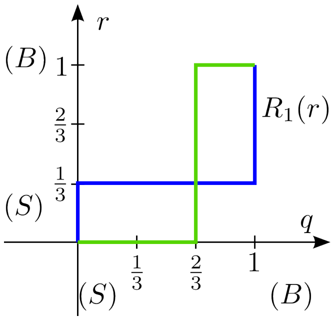
	Bach	Stravinsky
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

- Player 1's expected utility is $q(3r-1)+1-r$.
- Player 2's expected utility is $r(3q-2)+2(1-q)$.
- The best response functions are

$$R_1(r) = \begin{cases} 0 & \text{if } r < \frac{1}{3} \\ 1 & \text{if } r > \frac{1}{3} \\ [1,0] & \text{if } r = \frac{1}{3} \end{cases} \text{ and } R_2(q) = \begin{cases} 0 & \text{if } r < \frac{2}{3} \\ 1 & \text{if } r > \frac{2}{3} \\ [1,0] & \text{if } r = \frac{2}{3} \end{cases} .$$

BoS

- The two best response curves have three intersections!



- So there are three mixed-strategy Nash equilibria:
  - $(q^*, r^*) = (0,0)$, $(\frac{2}{3}, \frac{1}{3})$, and $(1,1)$.
  - Two of them are pure-strategy Nash equilibria: $(0,0)$ means both choosing S and $(1,1)$ means both choosing B.

Mixed strategies over more actions

- Consider the game "Rock, paper, scissor":

	R	P	S
R	$0,0$	$-1,1$	$1,-1$
P	$1,-1$	$0,0$	$-1,1$
S	$-1,1$	$1,-1$	$0,0$

- When a player has three actions, a mixed strategy is described with two variables.
  - E.g., player 1's mixed strategy is $(q_1, q_2, 1 - q_1 - q_2)$.
- When a player's action space is infinite (e.g., those players in the Cournot competition), a mixed strategy is a continuous probability distribution.

Existence of (mixed-strategy) Nash equilibrium

- In his work in 1950, John Nash proved the following theorem regarding the **existence** of Nash equilibrium:

Proposition 1

For a static game, if the number of players is finite and the action spaces are all finite, there exists at least one mixed-strategy Nash equilibrium.

  - This is a sufficient condition. Is it necessary?

Road map

- Introduction.
- Nash equilibrium.
- Mixed strategies.
- **Zero-sum games**.
- Zero-sum games and duality.

Zero-sum games

- For some games, one's **success** is the other one's **failure**.
  - When one earns $1, the other one loses $1.
- These games are called **zero-sum games**.
  - The sum of all players' payoffs are always zero under any action profile is zero.
- What is the optimal strategy in a zero-sum game?
  - One's optimal strategy is to **destroy** the other one.

Zero-sum games

- As an example, the following game is a zero-sum game:

	L	C	R
T	$4, -4$	$4, -4$	$10, -10$
M	$2, -2$	$3, -3$	$1, -1$
B	$6, -6$	$5, -5$	$7, -7$

- For a zero-sum game, we typically remove player 2's payoff:

	L	C	R
T	4	4	10
M	2	3	1
B	6	5	7

  - Player 1 wants to maximize her payoff.
  - Player 2 wants to minimize player 1's payoff.

Player 1's problem

- How to solve a zero-sum game?
  - The idea of Nash equilibrium still applies. However, the unique structure of zero-sum games allows us to solve them differently.
- Player 1 thinks:
  - If I choose T, he will choose L or C. I get 4.
  - If I choose M, he will choose R. I get 1.
  - If I choose B, he will choose C. I get 5.
- For each of player 1's actions, what he may get in equilibrium can only be the **row minimum**.

	L	C	R	Row min
T	4	4	10	4
M	2	3	1	1
B	6	5	7	5

Player 2's problem

- Player 2 thinks:
  - If I choose L, she will choose B. She get 6.
  - If I choose C, she will choose B. She get 5.
  - If I choose R, she will choose T. She get 10.
- For each of player 2's actions, what player 1 may get in equilibrium must be the **column maximum**.

	L	C	R	Row min
T	4	4	10	4
M	2	3	1	1
B	6	5	7	5
Column max	6	5	10	

- In equilibrium, player 1 **maximizes the row minimum** and player 2 minimizes the column maximum.
- The unique Nash equilibrium is (B, C).

Saddle points

- For a zero-sum game, a pure-strategy Nash equilibrium is called a **saddle point**.
- While there may not exist a pure-strategy Nash equilibrium for a general game, this also holds for a zero-sum game.
  - Any example?
- Is there any condition for a pure-strategy Nash equilibrium to exist in a zero-sum game?

Existence of saddle points

	L	C	R	R. min
T	4	4	10	4
M	2	3	1	2
B	6	5	7	5
C. max	6	5	10	

	H	T	R. min
H	1	-1	-1
T	-1	1	-1
C. max	1	1	

- For the previous example with a pure-strategy Nash equilibrium,

$$\max\{\text{row minima}\} = 5 = \min\{\text{column maxima}\}.$$

- For the zero-sum game matching penny with no pure-strategy Nash equilibrium,

$$\max\{\text{row minima}\} = 1 \neq -1 = \min\{\text{column maxima}\}.$$

Existence of saddle points

- Is there any relationship between the existence of saddle points and the values of max{row minima} and min{column maxima}?

Proposition 2

For a two-player zero-sum game, if

$$\max\{\textit{row minima}\} = \min\{\textit{column maxima}\},$$

an intersection of a $\max\{\textit{row minima}\}$ *and a* $\min\{\textit{column maxima}\}$ *is a saddle point.*

- To prove this, we rely on linear programming. In particular, we will apply **LP duality**.

Road map

- Introduction.
- Nash equilibrium.
- Mixed strategies.
- Zero-sum games.
- **Zero-sum games and duality**.

Mixed strategies for zero-sum games

- For a zero-sum game:
  - A pure-strategy Nash equilibrium (i.e., saddle point) may not exist.
  - A mixed-strategy Nash equilibrium must exist.
- How do players choose their mixed strategies?
- Recall that when a saddle point exists:
  - Player 1 chooses a row to maximize row minimum.
  - Player 2 chooses a column to minimize the column maximum.
- In general:
  - Player 1 chooses a row to maximize the **expectation** of row payoffs **under player 2's mixed strategy**.
  - Player 2 chooses a column to minimize the expectation of column payoffs under player 1's mixed strategy.

Mixed strategies for zero-sum games

- Suppose player 1's mixed strategy is $x = (x_1, x_2, x_3)$:

	L	C	R
T (with probability x_1)	4	4	10
M (with probability x_2)	2	3	1
B (with probability x_3)	6	5	7
Expected column payoff	$4x_1 + 2x_2 + 6x_3$	$4x_1 + 3x_2 + 5x_3$	$10x_1 + x_2 + 7x_3$

- Player 2 will find the smallest expected column maximum.
- Therefore, Player 1 should solve

$$\begin{aligned} \max \quad & \min\{4x_1 + 2x_2 + 6x_3, 4x_1 + 3x_2 + 5x_3, 10x_1 + x_2 + 7x_3\} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

Linearization of player 1's problem

$$\begin{aligned} \max \quad & \min\{4x_1 + 2x_2 + 6x_3, 4x_1 + 3x_2 + 5x_3, 10x_1 + x_2 + 7x_3\} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

- Player 1's problem is nonlinear.
- However, it is equivalent to the following linear program:

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v \leq 4x_1 + 2x_2 + 6x_3 \\ & v \leq 4x_1 + 3x_2 + 5x_3 \\ & v \leq 10x_1 + x_2 + 7x_3 \\ & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

Player 2's problem

- Suppose player 2's mixed strategy is $y = (y_1, y_2, y_3)$.
- Following the same logic, player 2 solves the linear program

$$
\begin{aligned}
\min \quad & u \\
\text{s.t.} \quad & u \geq 4y_1 + 4y_2 + 10y_3 \\
& u \geq 2y_1 + 3y_2 + y_3 \\
& u \geq 6y_1 + 5y_2 + 7y_3 \\
& y_1 + y_2 + y_3 = 1 \\
& y_i \geq 0 \quad \forall i = 1, ..., 3.
\end{aligned}
$$

Duality between the two players

- The two players' problems can be rewritten to

$$
\begin{array}{rrcrcrcrcl}
z^* = \max & & & & & & & v & & \\
\text{s.t.} & -4x_1 & - & 2x_2 & - & 6x_3 & + & v & \leq & 0 \\
 & -4x_1 & - & 3x_2 & - & 5x_3 & + & v & \leq & 0 \\
 & -10x_1 & - & x_2 & - & 7x_3 & + & v & \leq & 0 \\
 & x_1 & + & x_2 & + & x_3 & & & = & 1 \\
\end{array}
$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ v \text{ urs.}$$

$$
\begin{array}{rrcrcrcrcl}
w^* = \min & & & & & & & u & & \\
\text{s.t.} & -4y_1 & - & 4y_2 & - & 10y_3 & + & u & \geq & 0 \\
 & -2y_1 & - & 3y_2 & - & y_3 & + & u & \geq & 0 \\
 & -6y_1 & - & 5y_2 & - & 7y_3 & + & u & \geq & 0 \\
 & y_1 & + & y_2 & + & y_3 & & & = & 1 \\
\end{array}
$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ u \text{ urs.}$$

- This is a **primal-dual pair**!

Duality between the two players

- For a two-player zero-sum game, if an LP finds player 1's optimal strategy, its **dual** finds player 2's optimal strategy.
  - A pair of primal and dual optimal solutions x^* and y^* form a mixed-strategy Nash equilibrium.
- Some examples in business:
  - Two competing retailers sharing a fixed amount of consumers.
  - A retailer and a manufacturer negotiating the price of a product.
- Can any of these two LPs be infeasible or unbounded?
  - No! Because a mixed-strategy Nash equilibrium **always exists**.
  - So these two LPs must both have optimal solutions.

Existence of saddle points

- Now we are ready to prove the theorem regarding the existence of saddle points:

  For a two-player zero-sum game, if

  $$\max\{\textit{row minima}\} = \min\{\textit{column maxima}\},$$

  an intersection of a $\max\{\textit{row minima}\}$ *and a* $\min\{\textit{column maxima}\}$ *is a saddle point.*

Existence of saddle points

- First of all, note that choosing a single row or column corresponds to a feasible primal or dual solution:
  - Choosing a single row is for player 1 to implement a pure strategy (by setting the corresponding $x_i = 1$ and $x_k = 0$ for all $k \neq i$).
  - This is a feasible solution to the primal LP.
  - Similarly, choosing a single column corresponds to a feasible solution to the dual LP with $y_j = 1$ and $y_k = 0$ for all $k \neq j$.
- Suppose max{row minima} = min{column maxima} is satisfied:
  - Suppose this occurs at row i and column j.
  - Let x^* be the primal solution with $x_i^* = 1$ and $x_k^* = 0$ for all $k \neq i$.
  - Let y^* be the dual solution with $y_j^* = 1$ and $y_k^* = 0$ for all $k \neq j$.
  - As the condition is satisfied, $z^* = w^*$ for two feasible solutions. By strong duality, these two feasible solutions are both optimal.
- A pair of primal-dual optimal solutions form a mixed-strategy Nash equilibrium. As $x_i^* = y_j^* = 1$, x^* and y^* form a saddle point.