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Suggested Solution for Pre-lecture Problems for Lecture 12

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1. (a) By leading principal minors:

$$|2| = 2 \quad \text{and} \quad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Therefore, this matrix is positive semi-definite.

(b) By leading principal minors:

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1.$$

Therefore, this matrix is not positive semi-definite.

(c) By leading principal minors:

$$|1| = 1, \quad \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \quad \text{and} \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 6.$$

Therefore, this matrix is positive semi-definite.

2. (a)

$$f'(x) = 3x^2 + 4x + 1 \quad \text{and} \quad f''(x) = 6x + 4.$$

Therefore, $f(x)$ is convex for $x \in [-\frac{2}{3}, \infty)$.

(b)

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 + 1 \\ 4x_2 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 & 0 \\ 0 & 4 \end{bmatrix}.$$

By leading principal minors: $\nabla^2 f(x_1, x_2)$ is positive semi-definite iff

$$6x_1 \geq 0 \quad \text{and} \quad \begin{vmatrix} 6x_1 & 0 \\ 0 & 4 \end{vmatrix} = 12x_1 \geq 0.$$

Therefore, $f(x_1, x_2)$ is convex for $x_1 \in [-\frac{2}{3}, \infty)$, $x_2 \in \mathbb{R}$.

(c)

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1x_3 + 1 \\ 2x_3 \\ x_1^2 + 2x_2 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} 2x_3 & 0 & 2x_1 \\ 0 & 0 & 2 \\ 2x_1 & 2 & 0 \end{bmatrix}.$$

By leading principal minors: $\nabla^2 f(x_1, x_2, x_3)$ is positive semi-definite iff

$$2x_3 \geq 0, \quad \begin{vmatrix} 2x_3 & 0 \\ 0 & 0 \end{vmatrix} = 0 \geq 0 \quad \text{and} \quad \begin{vmatrix} 2x_3 & 0 & 2x_1 \\ 0 & 0 & 2 \\ 2x_1 & 2 & 0 \end{vmatrix} = -8x_3 \geq 0.$$

Therefore, $f(x_1, x_2, x_3)$ is convex for $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, and $x_3 = 0$.

3. (a) Let $f(x) = (x_1 - 2)^2 + (x_2 - 3)^2$.

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 - 4 \\ 2x_2 - 6 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

By leading principal minors:

$$|2| = 2 \quad \text{and} \quad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4.$$

Therefore, $f(x_1, x_2)$ is convex for $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$.
Since the feasible region of the NLP is convex and the objective function of the NLP over the feasible region is convex , the NLP is a convex program.

(b)

$$\mathcal{L}(x_1, x_2|\lambda) = (x_1 - 2)^2 + (x_2 - 3)^2 - \lambda(4 - 2x_1 - x_2), \text{ where } \lambda \geq 0.$$

(c) The Lagrangian relaxation is

$$z^L(\lambda) = \min \mathcal{L}(x_1, x_2|\lambda) = \min(x_1 - 2)^2 + (x_2 - 3)^2 - \lambda(4 - 2x_1 - x_2).$$

(d)

$$\nabla\mathcal{L} = 0 \Rightarrow \begin{bmatrix} 2(x_1 - 2) + 2\lambda \\ 2(x_2 - 3) + \lambda \end{bmatrix} = 0 \Rightarrow x_1 - 2 = 2x_2 - 6 \Rightarrow x_1 - 2x_2 + 4 = 0$$

(e) Based on the complementary slackness $\lambda(4 - 2x_1 - x_2) = 0$, either $\lambda = 0$ or $(4 - 2x_1 - x_2) = 0$. If $\lambda = 0$, the solution is $(x_1, x_2) = (2, 3)$, which is against the primal feasibility $2x_1 - x_2 \leq 4$. If $(4 - 2x_1 - x_2) = 0.$, the solution is $(x_1, x_2) = (0.8, 2.4)$, which is feasible and therefore optimal.