Operations Research, Spring 2017
Lecture 10: Multi-variate Nonlinear Programming

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (1 point each) Consider the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}, \quad \begin{bmatrix} -4 & 2 \\ 2 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Which of them are positive semi-definite?

2. (2 points each) For each of the following functions, determine the region over which the function is convex.

(a) $f(x_1, x_2) = x_1^2 + x_2^3$.

(b) $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2$.

(c) $f(x_1, x_2) = x_1^3 - x_1^2 x_2 + x_2^2$.

(d) $f(x_1, x_2, x_3) = x_1^2 + x_1 x_2 x_3$.

(e) $f(x_1, x_2, x_3) = -\sqrt{x_1} + \frac{1}{x_2} + e^{-x_3}$.

3. (2 points each) Consider the NLP

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + 2x_2^2 \leq 12. \end{aligned}$$

(a) Will the constraint always be binding at an optimal solution?

(b) Find the Lagrangian. What is the sign of your Lagrange multiplier?

(c) Formulate the Lagrangian relaxation.

(d) According to the FOC for the Lagrangian, what must be satisfied by an optimal solution?

(e) Find an optimal solution to the NLP.

4. Consider the NLP

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq b. \end{aligned}$$

(a) (2 points) Will the constraint always be binding at an optimal solution?

(b) (2 points) Find the Lagrangian. What is the sign of your Lagrange multiplier?

(c) (3 points) Let $b = 6$. Solve the NLP.

(d) (3 points) Let $b = -14$. Solve the NLP.

5. (2 points each) A retailer sells products 1 and 2 at prices p_1 and p_2. For product i, the demand is

$$q_i = a - p_i + bp_{3-i}, \quad i = 1, 2,$$

where $a > 0$ and $b \in [0, 1)$. The retailer sets p_1 and p_2 to maximize its total profit. Assume that there is no production cost.

(a) Explain why $b \in [0, 1)$ is reasonable.

(b) Formulate the retailer's problem.

(c) Is this a convex program?

(d) Solve the retailer's problem.

(e) How do the optimal prices change with a and b? Does that make sense?

6. (5 points each) Consider a set of data (x_i, y_i), $i = 1, ..., n$. If we believe that x_i and y_i has a linear relationship, we may apply simple linear regression to fit these data. More precisely, we try to find α and β such that the line $y = \alpha + \beta x$ minimizes the sum of squared errors for all the data points:

$$\min_{\alpha,\beta} \sum_{i=1}^{n} \left[y_i - (\alpha + \beta x_i) \right]^2$$

(a) Is this a convex program?

(b) Solve the NLP to derive the formula for simple linear regression.