Note on Chapter 3 of [Manber]: Solving a Recurrence Relation with Generating Functions

Generating Functions provide a systematic, effective means for representing and manipulating infinite sequences (of numbers). We use them here to derive a closed-form representation of the Fibonacci sequence as defined by the following recurrence relation:

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n = F_{n-2} + F_{n-1} & \text{for } n > 2 \end{cases}$$

Below are two basic generating functions; the second one is a generalization of the first and will be used in our solution.

generating function	power series	generated sequence
$\frac{1}{1-z}$	$1 + z + z^2 + z^3 + \cdots + z^n + \cdots$	$1, 1, 1, \cdots, 1, \cdots$
$\frac{c}{1-az}$	$c + caz + ca^2z^2 + ca^3z^3 + \cdots + ca^nz^n + \cdots$	$c, ca, ca^2, ca^3, \cdots, ca^n, \cdots$

Let $F(z) = 0 + F_1z + F_2z^2 + F_3z^3 + \cdots + F_nz^n + \cdots$ (a generating function for the Fibonacci sequence).

$$\begin{array}{rcl} F(z) & = & F_1z + F_2z^2 + F_3z^3 + \cdots + F_nz^n + F_{n+1}z^{n+1} + \cdots \\ zF(z) & = & F_1z^2 + F_2z^3 + \cdots + F_{n-1}z^n + F_nz^{n+1} + \cdots \\ z^2F(z) & = & F_1z^3 + F_2z^4 + \cdots + F_{n-2}z^n + F_{n-1}z^{n+1} + \cdots \\ \hline (1 - z - z^2)F(z) & = & z \end{array}$$

Continuing from $(1 - z - z^2)F(z) = z$,

$$\begin{array}{rcl} F(z) & = & \frac{z}{1-z-z^2} \quad (= \frac{z}{(1-\frac{1+\sqrt{5}}{2}z)(1-\frac{1-\sqrt{5}}{2}z)}) \\ & = & \frac{\frac{1}{\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} + \frac{-\frac{1}{\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z} \\ & = & (\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}z + \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^2z^2 + \cdots + \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^nz^n + \cdots) + \\ & & (-\frac{1}{\sqrt{5}} + (-\frac{1}{\sqrt{5}})\frac{1-\sqrt{5}}{2}z + (-\frac{1}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^2z^2 + \cdots + (-\frac{1}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^nz^n + \cdots) \\ & = & z + z^2 + \cdots + (\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n)z^n + \cdots \end{array}$$

Therefore, $F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.