

## Note on Chapter 3 of [Manber]: Solving a Recurrence Relation with Generating Functions

*Generating Functions* provide a systematic, effective means for representing and manipulating infinite sequences (of numbers). We use them here to derive a closed-form representation of the Fibonacci sequence as defined by the following recurrence relation:

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n = F_{n-2} + F_{n-1} \quad \text{for } n > 2 \end{cases}$$

Below are two basic generating functions; the second one is a generalization of the first and will be used in our solution.

generating function	power series	generated sequence
$\frac{1}{1-z}$	$1 + z + z^2 + z^3 + \cdots + z^n + \cdots$	$1, 1, 1, \dots, 1, \dots$
$\frac{c}{1-az}$	$c + caz + ca^2z^2 + ca^3z^3 + \cdots + ca^nz^n + \cdots$	$c, ca, ca^2, ca^3, \dots, ca^n, \dots$

Let  $F(z) = 0 + F_1z + F_2z^2 + F_3z^3 + \cdots + F_nz^n + \cdots$  (a generating function for the Fibonacci sequence).

$$\begin{aligned} F(z) &= F_1z + F_2z^2 + F_3z^3 + \cdots + F_nz^n + F_{n+1}z^{n+1} + \cdots \\ zF(z) &= F_1z^2 + F_2z^3 + \cdots + F_{n-1}z^n + F_nz^{n+1} + \cdots \\ z^2F(z) &= F_1z^3 + F_2z^4 + \cdots + F_{n-2}z^n + F_{n-1}z^{n+1} + \cdots \\ \hline (1 - z - z^2)F(z) &= z \end{aligned}$$

Continuing from  $(1 - z - z^2)F(z) = z$ ,

$$\begin{aligned} F(z) &= \frac{z}{1-z-z^2} \quad \left( = \frac{z}{(1-\frac{1+\sqrt{5}}{2}z)(1-\frac{1-\sqrt{5}}{2}z)} \right) \\ &= \frac{\frac{1}{\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} + \frac{-\frac{1}{\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z} \\ &= \left( \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}z + \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^2 z^2 + \cdots + \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n z^n + \cdots \right) + \\ &\quad \left( -\frac{1}{\sqrt{5}} + \left( -\frac{1}{\sqrt{5}} \right) \frac{1-\sqrt{5}}{2}z + \left( -\frac{1}{\sqrt{5}} \right) \left( \frac{1-\sqrt{5}}{2} \right)^2 z^2 + \cdots + \left( -\frac{1}{\sqrt{5}} \right) \left( \frac{1-\sqrt{5}}{2} \right)^n z^n + \cdots \right) \\ &= z + z^2 + \cdots + \left( \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \right) z^n + \cdots \end{aligned}$$

Therefore,  $F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$ .