Basic Graph Algorithms

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The Königsberg Bridges Problem

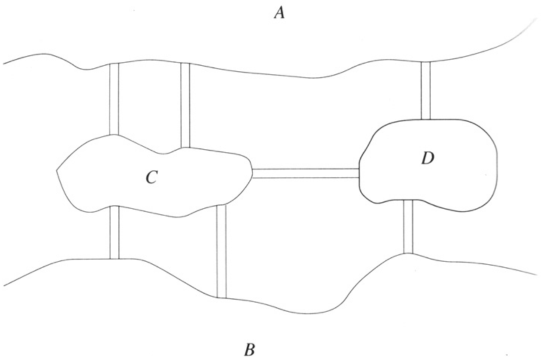


Figure 7.1 The Königsberg bridges problem.

Source: Manber 1989

Can one start from one of the lands, cross every bridge exactly once, and return to the origin?

The Königsberg Bridges Problem (cont.)

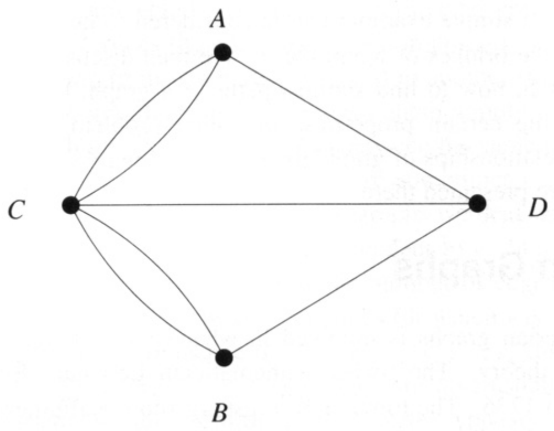


Figure 7.2 The graph corresponding to the Königsberg bridges problem.

Source: Manber 1989

Graphs

- A graph consists of a set of vertices (or nodes) and a set of edges (or links, each normally connecting two vertices).
- A graph is commonly denoted as $G(V, E)$, where
  - G is the name of the graph,
  - V is the set of vertices, and
  - E is the set of edges.

Modeling with Graphs

- Reachability
  - Finding program errors
  - Solving sliding tile puzzles
- Shortest Paths
  - Finding the fastest route to a place
  - Routing messages in networks
- Graph Coloring
  - Coloring maps
  - Scheduling classes

Graphs (cont.)

- Undirected vs. Directed Graph
- Simple Graph vs. Multigraph
- Path, Simple Path, Trail
- Circuit, Cycle
- Degree, In-Degree, Out-Degree
- Connected Graph, Connected Components
- Tree, Forest
- Subgraph, Induced Subgraph
- Spanning Tree, Spanning Forest
- Weighted Graph

Eulerian Graphs

Problem

Given an undirected connected graph $G = (V, E)$ such that all the vertices have even degrees, find a circuit P such that each edge of E appears in P exactly once.

The circuit P in the problem statement is called an *Eulerian circuit*.

Theorem

An undirected connected graph has an Eulerian circuit if and only if all of its vertices have even degrees.

Depth-First Search

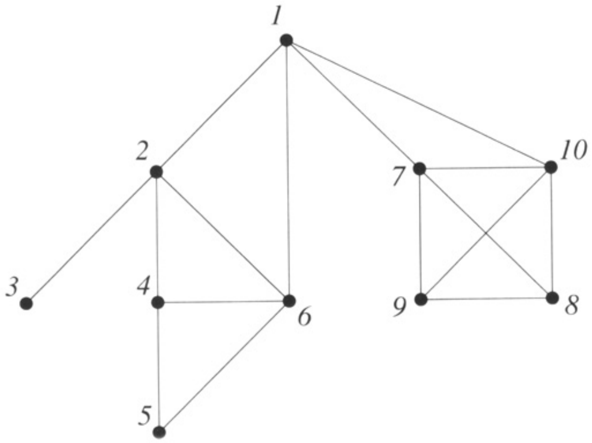


Figure 7.4 A DFS for an undirected graph.

Source: Manber 1989

Depth-First Search (cont.)

```
Algorithm Depth_First_Search($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            $Depth\_First\_Search(G, w)$;
        perform postWORK for $(v, w)$
end
```

Depth-First Search (cont.)

```
Algorithm Refined_DFS($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            Refined_DFS($G, w$);
        perform postWORK for $(v, w)$;
    perform postWORK_II on $v$
end
```

Connected Components

```
Algorithm Connected_Components($G$);
begin
    $Component\_Number := 1$;
    while there is an unmarked vertex $v$ do
        $Depth\_First\_Search(G, v)$
        (preWORK:
            $v.Component := Component\_Number$);
        $Component\_Number := Component\_Number + 1$
end
```

DFS Numbers

```
**Algorithm DFS_Numbering**$(G, v)$;
**begin**
    $DFS\_Number := 1$;
    $Depth\_First\_Search(G, v)$
    (preWORK:
        $v.DFS := DFS\_Number$;
        $DFS\_Number := DFS\_Number + 1$)
**end**
```

The DFS Tree

```
**Algorithm Build_DFS_Tree**$(G, v)$;
**begin**
    *Depth_First_Search*$(G, v)$
    (postWORK:
        **if** $w$ was unmarked **then**
            add the edge $(v, w)$ to $T$);
**end**
```

The DFS Tree (cont.)

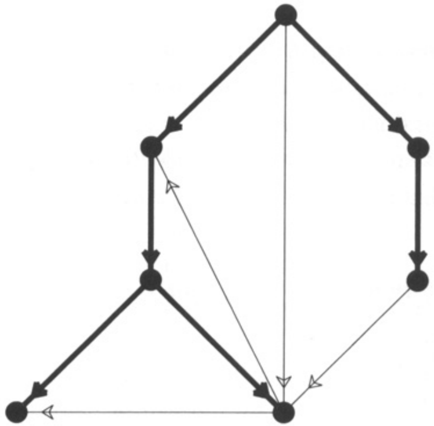


Figure 7.9 A DFS tree for a directed graph.

Source: Manber 1989

The DFS Tree (cont.)

Lemma (7.2)

For an undirected graph $G = (V, E)$, every edge $e \in E$ either belongs to the DFS tree T, or connects two vertices of G, one of which is the ancestor of the other in T.

For undirected graphs, DFS avoids cross edges.

Lemma (7.3)

For a directed graph $G = (V, E)$, if (v, w) is an edge in E such that $v.DFS\_Number < w.DFS\_Number$, then w is a descendant of v in the DFS tree T.

For directed graphs, cross edges must go "from right to left".

Directed Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle.

Lemma (7.4)

G contains a directed cycle if and only if G contains a back edge (relative to the DFS tree).

Directed Cycles (cont.)

```
**Algorithm Find_a_Cycle**($G$);
**begin**
    $Depth\_First\_Search(G, v)$ /* arbitrary $v$ */
    (preWORK:
        $v.on\_the\_path := true$;
     postWORK:
        **if** $w.on\_the\_path$ **then**
            $Find\_a\_Cycle := true$;
            halt;
        **if** $w$ is the last vertex on $v$'s list **then**
            $v.on\_the\_path := false$;)
**end**
```

Directed Cycles (cont.)

```
Algorithm Refined_Find_a_Cycle($G$);
begin
    $Refined\_DFS(G, v)$ /* arbitrary $v$ */
    (preWORK:
        $v.on\_the\_path := true$;
     postWORK:
        if $w.on\_the\_path$ then
            $Refined\_Find\_a\_Cycle := true$;
            halt;
     postWORK_II:
        $v.on\_the\_path := false$)
end
```

Breadth-First Search

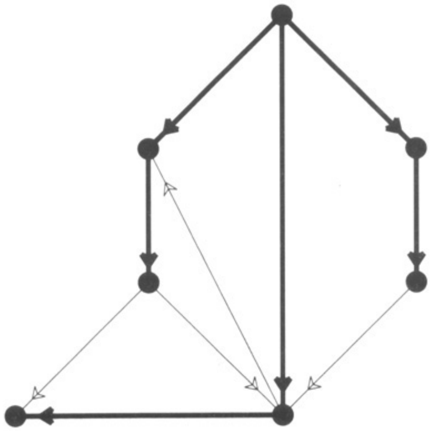


Figure 7.12 A BFS tree for a directed graph.

Source: Manber 1989

Breadth-First Search (cont.)

```
Algorithm Breadth_First_Search($G, v$);
begin
    mark $v$;
    put $v$ in a queue;
    while the queue is not empty do
        remove vertex $w$ from the queue;
        perform preWORK on $w$;
        for all edges $(w, x)$ with $x$ unmarked do
            mark $x$;
            add $(w, x)$ to the $BFS$ tree $T$;
            put $x$ in the queue
end
```

Breadth-First Search (cont.)

Lemma (7.5)

If an edge (u, w) belongs to a BFS tree such that u is a parent of w, then u has the minimal BFS number among vertices with edges leading to w.

Lemma (7.6)

For each vertex w, the path from the root to w in T is a shortest path from the root to w in G.

Lemma (7.7)

If an edge (v, w) in E does not belong to T and w is on a larger level, then the level numbers of w and v differ by at most 1.

Breadth-First Search (cont.)

```
Algorithm Simple_BFS($G, v$);
begin
    put $v$ in Queue;
    while Queue is not empty do
        remove vertex $w$ from Queue;
        if $w$ is unmarked then
            mark $w$;
            perform preWORK on $w$;
            for all edges $(w, x)$ with $x$ unmarked do
                put $x$ in Queue
end
```

Breadth-First Search (cont.)

```
Algorithm Simple_Nonrecursive_DFS($G, v$);
begin
    push $v$ to $Stack$;
    while $Stack$ is not empty do
        pop vertex $w$ from $Stack$;
        if $w$ is unmarked then
            mark $w$;
            perform preWORK on $w$;
            for all edges $(w, x)$ with $x$ unmarked do
                push $x$ to $Stack$
end
```

Topological Sorting

Problem

Given a directed acyclic graph $G = (V, E)$ with n vertices, label the vertices from 1 to n such that, if v is labeled k, then all vertices that can be reached from v by a directed path are labeled with labels $> k$.

Lemma (7.8)

A directed acyclic graph always contains a vertex with indegree 0.

Topological Sorting (cont.)

```
Algorithm Topological_Sorting($G$);
    initialize $v.indegree$ for all vertices; /* by $DFS$ */
    $G\_label := 0$;
    for $i := 1$ to $n$ do
        if $v_i.indegree = 0$ then put $v_i$ in $Queue$;
    repeat
        remove vertex $v$ from $Queue$;
        $G\_label := G\_label + 1$;
        $v.label := G\_label$;
        for all edges $(v, w)$ do
            $w.indegree := w.indegree - 1$;
            if $w.indegree = 0$ then put $w$ in $Queue$
    until $Queue$ is empty
```

Single-Source Shortest Paths

Problem

Given a directed graph $G = (V, E)$ and a vertex v, find shortest paths from v to all other vertices of G.

Shorted Paths: The Acyclic Case

```
**Algorithm Acyclic_Shortest_Paths**$(G, v, n)$;
{After performing a topological sort on $G$, ...}
**begin**
    let $z$ be the vertex labeled $n$;
    **if** $z \neq v$ **then**
        $Acyclic\_Shortest\_Paths(G - z, v, n - 1)$;
        **for** all $w$ such that $(w, z) \in E$ **do**
            **if** $w.SP + length(w, z) < z.SP$ **then**
                $z.SP := w.SP + length(w, z)$
    **else** $v.SP := 0$
**end**
```

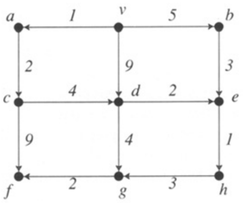
The Acyclic Case (cont.)

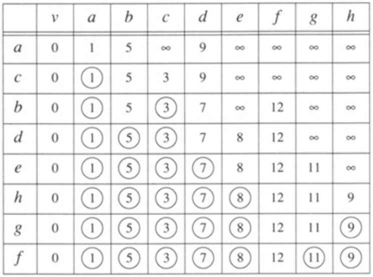
```
**Algorithm Imp_Acyclic_Shortest_Paths**$(G, v)$;
    **for** all vertices $w$ **do** $w.SP := \infty$;
    initialize $v.indegree$ for all vertices;
    **for** $i := 1$ to $n$ **do**
        **if** $v_i.indegree = 0$ **then** put $v_i$ in $Queue$;
    $v.SP := 0$;
    **repeat**
        remove vertex $w$ from $Queue$;
        **for** all edges $(w, z)$ **do**
            **if** $w.SP + length(w, z) < z.SP$ **then**
                $z.SP := w.SP + length(w, z)$;
            $z.indegree := z.indegree - 1$;
            **if** $z.indegree = 0$ **then** put $z$ in $Queue$
    **until** $Queue$ is empty
```

Shortest Paths: The General Case

```
**Algorithm Single_Source_Shortest_Paths**$(G, v)$;
**begin**
    **for** all vertices $w$ **do**
        $w.mark := false$;
        $w.SP := \infty$;
    $v.SP := 0$;
    **while** there exists an unmarked vertex **do**
        let $w$ be an unmarked vertex s.t. $w.SP$ is minimal;
        $w.mark := true$;
        **for** all edges $(w, z)$ such that $z$ is unmarked **do**
            **if** $w.SP + length(w, z) < z.SP$ **then**
                $z.SP := w.SP + length(w, z)$
**end**
```

The General Case (cont.)





	v	a	b	c	d	e	f	g	h
a	0	1	5	∞	9	∞	∞	∞	∞
c	0	①	5	3	9	∞	∞	∞	∞
b	0	①	5	③	7	∞	12	∞	∞
d	0	①	⑤	③	7	8	12	∞	∞
e	0	①	⑤	③	⑦	8	12	11	∞
h	0	①	⑤	③	⑦	⑧	12	11	9
g	0	①	⑤	③	⑦	⑧	12	11	⑨
f	0	①	⑤	③	⑦	⑧	12	⑪	⑨

Figure 7.18 An example of the single-source shortest-paths algorithm.

Source: Manber 1989

Minimum-Weight Spanning Trees

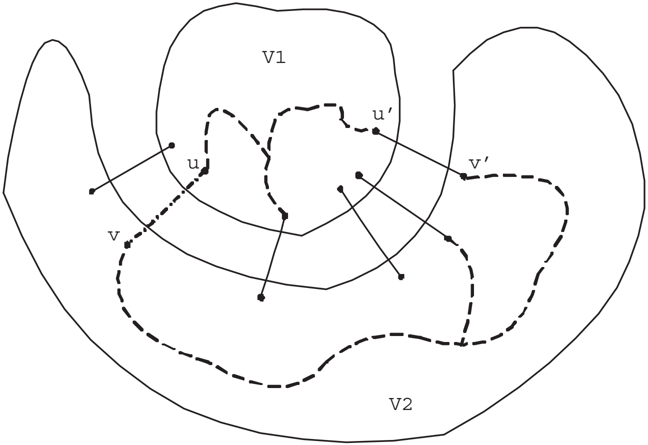
Problem

Given an undirected connected weighted graph $G = (V, E)$, find a spanning tree T of G of minimum weight.

Theorem

Let V_1 and V_2 be a partition of V and $E(V_1, V_2)$ be the set of edges connecting nodes in V_1 to nodes in V_2. The edge with the minimum weight in $E(V_1, V_2)$ must be in the minimum-cost spanning tree of G.

Minimum-Weight Spanning Trees (cont.)



If $cost(u, v)$ is the smallest among $E(V_1, V_2)$, then $\{u, v\}$ must be in the minimum spanning tree.

Minimum-Weight Spanning Trees (cont.)

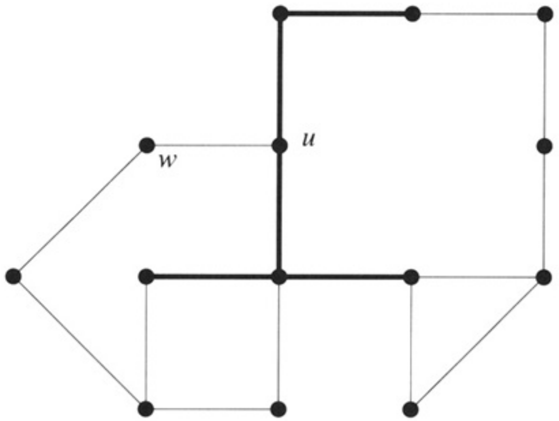


Figure 7.19 Finding the next edge of the MCST.

Source: Manber 1989

Minimum-Weight Spanning Trees (cont.)

```
Algorithm MST(G);
begin
    initially T is the empty set;
    for all vertices w do
        w.mark := false;  w.cost := ∞;
    let (x, y) be a minimum cost edge in G;
    x.mark := true;
    for all edges (x, z) do
        z.edge := (x, z);  z.cost := cost(x, z);
```

Minimum-Weight Spanning Trees (cont.)

```
    **while** there exists an unmarked vertex **do**
        let $w$ be an unmarked vertex with minimal $w.cost$;
        **if** $w.cost = \infty$ **then**
            print "G is not connected";  halt
        **else**
            $w.mark := true$;
            add $w.edge$ to $T$;
            **for** all edges $(w, z)$ **do**
                **if** not $z.mark$ **then**
                    **if** $cost(w, z) < z.cost$ **then**
                        $z.edge := (w, z)$;  $z.cost := cost(w, z)$
**end**
```

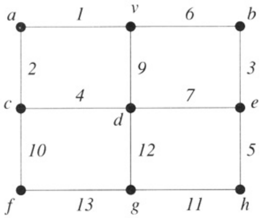
Minimum-Weight Spanning Trees (cont.)

```
Algorithm Another_MST($G$);
begin
    initially $T$ is the empty set;
    for all vertices $w$ do
        $w.mark := false$;  $w.cost := \infty$;
    $x.mark := true$; /* $x$ is an arbitrary vertex */
    for all edges $(x, z)$ do
        $z.edge := (x, z)$;  $z.cost := cost(x, z)$;
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected"; halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;
                        $z.cost := cost(w, z)$
end
```

Minimum-Weight Spanning Trees (cont.)



	v	a	b	c	d	e	f	g	h
v	-	$v(1)$	$v(6)$	∞	$v(9)$	∞	∞	∞	∞
a	-	-	$v(6)$	$a(2)$	$v(9)$	∞	∞	∞	∞
c	-	-	$v(6)$	-	$c(4)$	∞	$c(10)$	∞	∞
d	-	-	$v(6)$	-	-	$d(7)$	$c(10)$	$d(12)$	∞
b	-	-	-	-	-	$b(3)$	$c(10)$	$d(12)$	∞
e	-	-	-	-	-	-	$c(10)$	$d(12)$	$e(5)$
h	-	-	-	-	-	-	$c(10)$	$h(11)$	-
f	-	-	-	-	-	-	-	$h(11)$	-
g	-	-	-	-	-	-	-	-	-

Figure 7.21 An example of the minimum-cost spanning-tree algorithm.

Source: Manber 1989

All Shortest Paths

Problem

Given a weighted graph $G = (V, E)$ (directed or undirected) with nonnegative weights, find the minimum-length paths between all pairs of vertices.

```
Algorithm All_Pairs_Shortest_Paths($W$);
begin
    {initialization omitted}
    for $m := 1$ to $n$ do  {the induction sequence}
        for $x := 1$ to $n$ do
            for $y := 1$ to $n$ do
                if $W[x, m] + W[m, y] < W[x, y]$ then
                    $W[x, y] := W[x, m] + W[m, y]$
end
```

Transitive Closure

Problem

Given a directed graph $G = (V, E)$, find its transitive closure.

```
Algorithm Transitive_Closure($A$);
begin
    {initialization omitted}
    for $m := 1$ to $n$ do
        for $x := 1$ to $n$ do
            for $y := 1$ to $n$ do
                if $A[x, m]$ and $A[m, y]$ then
                    $A[x, y] := true$
end
```

Transitive Closure (cont.)

```
Algorithm Improved_Transitive_Closure($A$);
begin
    {initialization omitted}
    for $m$ := 1 to $n$ do
        for $x$ := 1 to $n$ do
            if $A[x, m]$ then
                for $y$ := 1 to $n$ do
                    if $A[m, y]$ then
                        $A[x, y]$ := $true$
end
```