Suggested Solutions to HW #1

2. (2.3) Find the following sum and prove your claim (i.e., guess and verify by induction):

$$1 \times 2 + 2 \times 3 + \cdots + n(n+1).$$

Solution. (Jen-Feng Shih)

From $\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$, we expect $F(n) = an^3 + bn^2 + cn + d$.

$F(1) = 2 = a + b + c + d$

$F(2) = 8 = 8a + 4b + 2c + d$

$F(3) = 20 = 27a + 9b + 3c + d$

$F(4) = 40 = 64a + 16b + 4c + d$

$a = \frac{1}{3}, b = 1, c = \frac{2}{3}, d = 0$

$F(n) = \frac{n^3+3n^2+2n}{3} = \frac{n(n+1)(n+2)}{3}$

Claim $F(n) = \frac{n(n+1)(n+2)}{3}$.

Base case: $F(1) = \frac{1(1+1)(1+2)}{3} = 2 = 1 \times 2$. The proof is by induction on n.

Inductive step:

$F(n+1)$

$= 1 \times 2 + 2 \times 3 + \cdots + n(n+1) + (n+1)(n+2)$

$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$ (from the Ind. Hypo.)

$= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$

$= \frac{(n+1)(n+2)(n+3)}{3}$

□

3. The Harmonic series $H(k)$ is defined by $H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}$. Prove that $H(2^n) \geq 1 + \frac{n}{2}$, for all $n \geq 0$ (which implies that $H(k)$ diverges).

Solution. (Jen-Feng Shih)

Base case: $H(2^0) = H(1) = 1 \geq 1 + \frac{0}{2}$. The proof is by induction on n.

Inductive step:

$H(2^{n+1})$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}}$

$= H(2^n) + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}}$

$\geq H(2^n) + \frac{2^n}{2^{n+1}}$

$\geq (1 + \frac{n}{2}) + \frac{1}{2}$, (from the Ind. Hypo.)

$= 1 + \frac{n+1}{2}$

□