Suggested Solutions to HW #3

1. (3.4) Below is a theorem from Manber's book:

> For all constants $c > 0$ and $a > 1$, and for all monotonically increasing functions $f(n)$, we have $(f(n))^c = O(a^{f(n)})$.

Prove, by using the above theorem, that for all constants $a, b > 0$, $(\log_2 n)^a = O(n^b)$.

Solution.(Jen-Feng Shih)
To avoid confusion in the variable names, we rename the variables and prove that for all constants $d, e > 0$, $(\log_2 n)^d = O(n^e)$.
Applying the theorem with $c = d > 0$, $a = 2^e > 1$, and $f(n) = \log_2 n$, we have
$(\log_2 n)^d$
$= O(a^{f(n)})$
$= O((2^e)^{\log_2 n})$
$= O(2^{e \times \log_2 n})$
$= O(2^{\log_2 n^e})$
$= O(n^e)$
□

4. (3.18) Consider the recurrence relation

$T(n) = 2T(n/2) + 1$, $T(2) = 1$.

We try to prove that $T(n) = O(n)$ (we limit our attention to powers of 2). We guess that $T(n) \leq cn$ for some (as yet unknown) c, and substitute cn in the expression. We have to show that $cn \geq 2c(n/2) + 1$. But this is clearly not true. Find the correct solution of this recurrence (you can assume that n is a power of 2), and explain why this attempt failed.

Solution.(Jinn-Shu Chang)
The attempt in this question failed because, in the case of a linear bound, a (negative) constant has to be included in the upper bound to cancel out the constant (1 in this case) in the recurrence relation.
Let us try a better guess: $T(n) \leq c(n-1)$. Substituting the upper bound $c((n/2)-1)$ for $T(n/2)$ in the induction step, we get

$T(n) = 2T(n/2) + 1$

$$= 2(c(n/2) - 1) + 1$$
$$= cn - 2 + 1$$
$$= cn - 1.$$

$c = 1$ will make $cn - 1$ less than or equal to $c(n-1)$. Hence we have proven that $T(n) \leq n - 1$. Since $n - 1 \leq n$, we have also proven that $T(n) \leq n$, implying $T(n) = O(n)$.
□