Note on Chapter 3 of [Manber]:
Solving a Recurrence Relation with Generating Functions

Generating Functions provide a systematic, effective means for representing and manipulating infinite sequences (of numbers). We use them here to derive a closed-form representation of the Fibonacci sequence as defined by the following recurrence relation:

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n = F_{n-2} + F_{n-1} \quad \text{for } n > 2 \end{cases}$$

Below are two basic generating functions; the second one is a generalization of the first and will be used in our solution.

generating function	power series	generated sequence
$\frac{1}{1-z}$	$1 + z + z^2 + z^3 + \cdots + z^n + \cdots$	$1, 1, 1, \cdots, 1, \cdots$
$\frac{c}{1-az}$	$c + caz + ca^2 z^2 + ca^3 z^3 + \cdots + ca^n z^n + \cdots$	$c, ca, ca^2, ca^3, \cdots, ca^n, \cdots$

Let $F(z) = 0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots + F_n z^n + \cdots$ (a generating function for the Fibonacci sequence).

$$\begin{aligned}
F(z) &= F_1 z + F_2 z^2 + F_3 z^3 + \cdots + F_n z^n + F_{n+1} z^{n+1} + \cdots \\
zF(z) &= F_1 z^2 + F_2 z^3 + \cdots + F_{n-1} z^n + F_n z^{n+1} + \cdots \\
z^2 F(z) &= F_1 z^3 + F_2 z^4 + \cdots + F_{n-2} z^n + F_{n-1} z^{n+1} + \cdots \\
\hline
(1 - z - z^2) F(z) &= z
\end{aligned}$$

Continuing from $(1 - z - z^2)F(z) = z$,

$$\begin{aligned}
F(z) &= \tfrac{z}{1-z-z^2} \quad \left(= \tfrac{z}{(1-\frac{1+\sqrt{5}}{2}z)(1-\frac{1-\sqrt{5}}{2}z)}\right) \\
&= \tfrac{\frac{1}{\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} + \tfrac{-\frac{1}{\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z} \\
&= \left(\tfrac{1}{\sqrt{5}} + \tfrac{1}{\sqrt{5}}\tfrac{1+\sqrt{5}}{2}z + \tfrac{1}{\sqrt{5}}(\tfrac{1+\sqrt{5}}{2})^2 z^2 + \cdots + \tfrac{1}{\sqrt{5}}(\tfrac{1+\sqrt{5}}{2})^n z^n + \cdots\right) + \\
&\quad \left(-\tfrac{1}{\sqrt{5}} + (-\tfrac{1}{\sqrt{5}})\tfrac{1-\sqrt{5}}{2}z + (-\tfrac{1}{\sqrt{5}})(\tfrac{1-\sqrt{5}}{2})^2 z^2 + \cdots + (-\tfrac{1}{\sqrt{5}})(\tfrac{1-\sqrt{5}}{2})^n z^n + \cdots\right) \\
&= z + z^2 + \cdots + \left(\tfrac{1}{\sqrt{5}}(\tfrac{1+\sqrt{5}}{2})^n - \tfrac{1}{\sqrt{5}}(\tfrac{1-\sqrt{5}}{2})^n\right) z^n + \cdots
\end{aligned}$$

Therefore, $F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.