Suggested Solutions to HW #1

2. The Harmonic series $H(k)$ is defined by $H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}$. Prove that $H(2^n) \geq 1 + \frac{n}{2}$, for all $n \geq 0$ (which implies that $H(k)$ diverges).

Solution. (Jen-Feng Shih)

The proof is by induction on n.

Base case: $H(2^0) = H(1) = 1 \geq 1 + \frac{0}{2}$.

Inductive step:

$H(2^{n+1})$
$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}}$
$= H(2^n) + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}}$
$\geq H(2^n) + \frac{2^n}{2^{n+1}}$
$\geq (1 + \frac{n}{2}) + \frac{1}{2}$ (from the Ind. Hypo.)
$= 1 + \frac{n+1}{2}$

□

3. (2.14) Consider binary trees where every internal node has two children. For any such tree T, let l_T denote the number of its leaves and m_T the number of its internal nodes. Prove *by induction* that $l_T = m_T + 1$.

Solution.

The proof is by strong induction on the number n_T of nodes of an arbitrary binary tree T where every internal nodes has two children.

Base case: $n_T = 1$. $l_T = 1$ and $m_T = 0$. Apparently, $l_T = m_T + 1$.

Inductive step: Assume $k \geq 1$ and $l_T = m_T + 1$ for all binary trees T with $1 \leq n_T \leq k$. Consider a binary tree T with $n_T = k + 1$. Let T_1 and T_2 denote respectively the left and the right subtrees of $T's$ root (which is an internal node and hence has two children). It is clear that every internal node of T_1 and T_2 has two children, $1 \leq n_{T_1} \leq k$, and $1 \leq n_{T_2} \leq k$. From the induction hypothesis, $l_{T_1} = m_{T_1} + 1$ and $l_{T_2} = m_{T_2} + 1$. The leaves of T_1 and T_2 are also leaves of T and the internal nodes of T_1 and T_2 are also internal nodes of T; therefore, $l_T = l_{T_1} + l_{T_2}$ and $m_T = m_{T_1} + m_{T_1} + 1$ (plus one for the root of T). It follows that

$$\begin{aligned} l_T &= l_{T_1} + l_{T_2} \\ &= (m_{T_1} + 1) + (m_{T_2} + 1) \\ &= (m_{T_1} + m_{T_2} + 1) + 1 \\ &= m_T + 1 \end{aligned}$$

□