


# Analysis of Algorithms

(Based on [Manber 1989])

Yih-Kuen Tsay

Department of Information Management  
National Taiwan University

# Introduction

-  The purpose of algorithm analysis is to **predict the behavior** (running time, space requirement, etc.) of an algorithm *without implementing it* on a specific computer. (Why?)


# Introduction

- 🌐 The purpose of algorithm analysis is to **predict the behavior** (running time, space requirement, etc.) of an algorithm *without implementing it* on a specific computer. (Why?)
- 🌐 As the exact behavior of an algorithm is hard to predict, the analysis is usually an *approximation*:
  - ☀ **Relative to the input size** (usually denoted by  $n$ ): input possibilities too enormous to elaborate
  - ☀ **Asymptotic**: should care more about larger inputs
  - ☀ **Worst-Case**: easier to do, often representative (Why not average-case?)
- 🌐 Such an approximation is usually good enough for **comparing** different algorithms for the same problem.

# Complexity

- 🌐 Theoretically, “complexity of an algorithm” is a more precise term for “approximate behavior of an algorithm”.
- 🌐 Two most important measures of complexity:
  - ☀️ **Time Complexity**  
an upper bound on the number of steps that the algorithm performs.
  - ☀️ **Space Complexity**  
an upper bound on the amount of temporary storage required for running the algorithm (excluding the input, the output, and the program itself).
- 🌐 We will focus on time complexity.

# Comparing Running Times

 How do we compare the following running times?

1.  $100n$
2.  $2n^2 + 50$
3.  $100n^{1.8}$

# Comparing Running Times

- How do we compare the following running times?
  - $100n$
  - $2n^2 + 50$
  - $100n^{1.8}$
- We will study an approach (the  $O$  notation) that allows us to ignore constant factors and concentrate on the behavior *as  $n$  goes to infinity*.
- For most algorithms, the constants in the expressions of their running times tend to be small.

# The $O$ Notation

- 🌐 A function  $g(n)$  is  $O(f(n))$  for another function  $f(n)$  if there exist constants  $c$  and  $N$  such that, for all  $n \geq N$ ,  $g(n) \leq cf(n)$ .
- 🌐 The function  $g(n)$  may be substantially less than  $cf(n)$ ; the  $O$  notation bounds it *only from above*.
- 🌐 The  $O$  notation allows us to **ignore constants** conveniently.

# The $O$ Notation

- 🌐 A function  $g(n)$  is  $O(f(n))$  for another function  $f(n)$  if there exist constants  $c$  and  $N$  such that, for all  $n \geq N$ ,  $g(n) \leq cf(n)$ .
- 🌐 The function  $g(n)$  may be substantially less than  $cf(n)$ ; the  $O$  notation bounds it *only from above*.
- 🌐 The  $O$  notation allows us to **ignore constants** conveniently.
- 🌐 Examples:
  - ☀  $5n^2 + 15 = O(n^2)$ .  
(cf.  $5n^2 + 15 \leq O(n^2)$  or  $5n^2 + 15 \in O(n^2)$ )
  - ☀  $5n^2 + 15 = O(n^3)$ .  
(cf.  $5n^2 + 15 \leq O(n^3)$  or  $5n^2 + 15 \in O(n^3)$ )
  - ☀ In an expression,  $T(n) = 3n^2 + O(n)$ .



# The $O$ Notation (cont.)

- 🌐 No need to specify the base of a logarithm.
  - ☀  $\log_2 n = \frac{\log_{10} n}{\log_{10} 2} = \frac{1}{\log_{10} 2} \log_{10} n$ .
  - ☀ For example, we can just write  $O(\log n)$ .
- 🌐  $O(1)$  denotes a constant.


# Properties of $O$

 We can add and multiply with  $O$ .

## Lemma (3.2)

1. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) + g(n) = O(s(n) + r(n))$ .
2. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) \cdot g(n) = O(s(n) \cdot r(n))$ .

# Properties of $O$

 We can add and multiply with  $O$ .

## Lemma (3.2)

1. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) + g(n) = O(s(n) + r(n))$ .
2. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) \cdot g(n) = O(s(n) \cdot r(n))$ .

 However, we cannot subtract or divide with  $O$ . (Why?)

# Properties of $O$

🌐 We can add and multiply with  $O$ .

## Lemma (3.2)

1. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) + g(n) = O(s(n) + r(n))$ .
2. If  $f(n) = O(s(n))$  and  $g(n) = O(r(n))$ , then  $f(n) \cdot g(n) = O(s(n) \cdot r(n))$ .

🌐 However, we cannot subtract or divide with  $O$ . (Why?)

- ☀️  $2n = O(n)$ ,  $n = O(n)$ , and  $2n - n = n \neq O(n - n)$ .
- ☀️  $n^2 = O(n^2)$ ,  $n = O(n^2)$ , and  $n^2/n = n \neq O(n^2/n^2)$ .

# Polynomial vs. Exponential

- 🌐 A function  $f(n)$  is *monotonically growing* if  $n_1 \geq n_2$  implies that  $f(n_1) \geq f(n_2)$ .
- 🌐 An exponential function grows *at least* as fast as a polynomial function does.

## Theorem (3.1)

*For all constants  $c > 0$  and  $a > 1$ , and for all monotonically growing functions  $f(n)$ ,  $(f(n))^c = O(a^{f(n)})$ .*

# Polynomial vs. Exponential

- 🌍 A function  $f(n)$  is *monotonically growing* if  $n_1 \geq n_2$  implies that  $f(n_1) \geq f(n_2)$ .
- 🌍 An exponential function grows *at least* as fast as a polynomial function does.

## Theorem (3.1)

For all constants  $c > 0$  and  $a > 1$ , and for all monotonically growing functions  $f(n)$ ,  $(f(n))^c = O(a^{f(n)})$ .

### 🌍 Examples:

- ☀️ Take  $n$  as  $f(n)$ ,  $n^c = O(a^n)$ .
- ☀️ Take  $\log_a n$  as  $f(n)$ ,  $(\log_a n)^c = O(a^{\log_a n}) = O(n)$ .

---

$\log n$	$n$	$n \log n$	$n^2$	$n^3$	$2^n$
0	1	0	1	1	2
1	2	2	4	8	4
2	4	8	16	64	16
3	8	24	64	512	256
4	16	64	256	4,096	65,536
5	32	160	1,024	32,768	4,294,967,296

---

**Table 1.7** Function values

Source: [E. Horowitz *et al.* 1998].

# Speed of Growth (cont.)

	<i>time 1</i>	<i>time 2</i>	<i>time 3</i>	<i>time 4</i>
running times	1000 steps/sec	2000 steps/sec	4000 steps/sec	8000 steps/sec
$\log_2 n$	0.010	0.005	0.003	0.001
$n$	1	0.5	0.25	0.125
$n \log_2 n$	10	5	2.5	1.25
$n^{1.5}$	32	16	8	4
$n^2$	1,000	500	250	125
$n^3$	1,000,000	500,000	250,000	125,000
$1.1^n$	$10^{39}$	$10^{39}$	$10^{38}$	$10^{38}$

**Table 3.1** Running times (in seconds) under different assumptions ( $n = 1000$ )

Source: [Manber 1989].



# $O$ , $o$ , $\Omega$ , and $\Theta$

- Let  $T(n)$  be the number of steps required to solve a given problem for input size  $n$ .
- We say that  $T(n) = \Omega(g(n))$  or the problem has a lower bound of  $\Omega(g(n))$  if there exist constants  $c$  and  $N$  such that, for all  $n \geq N$ ,  $T(n) \geq cg(n)$ .
- If a certain function  $f(n)$  satisfies both  $f(n) = O(g(n))$  and  $f(n) = \Omega(g(n))$ , then we say that  $f(n) = \Theta(g(n))$ .

# $O$ , $o$ , $\Omega$ , and $\Theta$

- Let  $T(n)$  be the number of steps required to solve a given problem for input size  $n$ .
- We say that  $T(n) = \Omega(g(n))$  or the problem has a lower bound of  $\Omega(g(n))$  if there exist constants  $c$  and  $N$  such that, for all  $n \geq N$ ,  $T(n) \geq cg(n)$ .
- If a certain function  $f(n)$  satisfies both  $f(n) = O(g(n))$  and  $f(n) = \Omega(g(n))$ , then we say that  $f(n) = \Theta(g(n))$ .
- We say that  $f(n) = o(g(n))$  if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

# Polynomial vs. Exponential (cont.)

- An exponential function grows *faster* than a polynomial function does.

## Theorem (3.3)

For all constants  $c > 0$  and  $a > 1$ , and for all monotonically growing functions  $f(n)$ , we have

$$(f(n))^c = o(a^{f(n)}).$$

- Consider a previous example again:  
Take  $\log_a n$  as  $f(n)$ . For all  $c > 0$  and  $a > 1$ ,

$$(\log_a n)^c = o(a^{\log_a n}) = o(n).$$

# Sums

- Techniques for summing expressions are essential for complexity analysis.
- For example, given that we know

$$S_0(n) = \sum_{i=1}^n 1 = n$$

and

$$S_1(n) = \sum_{i=1}^n i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

we want to compute the sum

$$S_2(n) = \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

# Sums (cont.)

From

$$(i + 1)^3 = i^3 + 3i^2 + 3i + 1,$$

we have

$$(i + 1)^3 - i^3 = 3i^2 + 3i + 1.$$

$$2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

$$3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$$

$$4^3 - 3^3 = 3 \times 3^2 + 3 \times 3 + 1$$

... ..

$$(n + 1)^3 - n^3 = 3 \times n^2 + 3 \times n + 1$$

---

$$(n + 1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n)$$

$$(S_3(n + 1) - S_3(1)) - S_3(n) = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n)$$

## Sums (cont.)

🌐 So, we have

$$(n + 1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n).$$

- 🌐 Given  $S_0(n)$  and  $S_1(n)$ , the sum  $S_2(n)$  can be computed by straightforward algebra.
- 🌐 Recall that the left-hand side  $(n + 1)^3 - 1$  equals  $(S_3(n + 1) - S_3(1)) - S_3(n)$ , a result from “**shifting and canceling**” terms of two sums.

## Sums (cont.)

🌐 So, we have

$$(n + 1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n).$$

- 🌐 Given  $S_0(n)$  and  $S_1(n)$ , the sum  $S_2(n)$  can be computed by straightforward algebra.
- 🌐 Recall that the left-hand side  $(n + 1)^3 - 1$  equals  $(S_3(n + 1) - S_3(1)) - S_3(n)$ , a result from “shifting and canceling” terms of two sums.
- 🌐 This generalizes to  $S_k(n)$ , for  $k > 2$ .
- 🌐 Similar shifting and canceling techniques apply to other kinds of sums.

# Recurrence Relations

- 🌐 A *recurrence relation* is a way to define a function by an expression involving the same function.
- 🌐 The Fibonacci numbers can be defined as follows:

$$\begin{cases} F(1) = 1 \\ F(2) = 1 \\ F(n) = F(n-2) + F(n-1) \end{cases}$$

We would need  $k - 2$  steps to compute  $F(k)$ .



# Recurrence Relations

- A *recurrence relation* is a way to define a function by an expression involving the same function.
- The Fibonacci numbers can be defined as follows:

$$\begin{cases} F(1) = 1 \\ F(2) = 1 \\ F(n) = F(n-2) + F(n-1) \end{cases}$$

We would need  $k - 2$  steps to compute  $F(k)$ .

- It is more convenient to have an explicit (or **closed-form**) expression.
- To obtain the explicit expression is called *solving* the recurrence relation.

# Guessing and Proving an Upper Bound

- 🌐 Recurrence relation: 
$$\begin{cases} T(2) = 1 \\ T(2n) \leq 2T(n) + 2n - 1 \end{cases}$$
- 🌐 Guess:  $T(n) = O(n \log n)$ .

# Guessing and Proving an Upper Bound

🌐 Recurrence relation: 
$$\begin{cases} T(2) = 1 \\ T(2n) \leq 2T(n) + 2n - 1 \end{cases}$$

🌐 Guess:  $T(n) = O(n \log n)$ .

🌐 Proof:

1. Base case:  $T(2) \leq 2 \log 2$ .

2. Inductive step: 
$$\begin{aligned} T(2n) &\leq 2T(n) + 2n - 1 \\ &\leq 2(n \log n) + 2n - 1 \\ &= 2n \log n + 2n \log 2 - 1 \\ &\leq 2n(\log n + \log 2) \\ &= 2n \log 2n \end{aligned}$$

# Recurrent Relations with Full History

Example:

$$T(n) = c + \sum_{i=1}^{n-1} T(i),$$

where  $c$  is a constant and  $T(1)$  is given separately.

$T(n) - T(n-1) = (c + \sum_{i=1}^{n-1} T(i)) - (c + \sum_{i=1}^{n-2} T(i)) = T(n-1)$ ;  
hence,  $T(n) = 2T(n-1)$ . (This holds only for  $n \geq 3$ .)


So, we get

$$\begin{cases} T(2) = c + T(1) \\ T(n) = 2T(n-1) \quad \text{if } n \geq 3 \end{cases}$$

which is easier to solve.




Other examples as a reading assignment ...

# Divide and Conquer Relations

-  The running time  $T(n)$  of a divide-and-conquer algorithm satisfies

$$T(n) = aT(n/b) + cn^k$$

where

-   $a$  is the number of subproblems,
-   $n/b$  is the size of each subproblem, and
-   $cn^k$  is the running time of the solutions-combining algorithm.

## Divide and Conquer Relations (cont.)

Assume, for simplicity,  $n = b^m$  ( $\frac{n}{b^m} = 1$ ,  $\frac{n}{b^{m-1}} = b$ , etc.).

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + cn^k \\ &= a\left(aT\left(\frac{n}{b^2}\right) + c\left(\frac{n}{b}\right)^k\right) + cn^k \\ &= a\left(a\left(aT\left(\frac{n}{b^3}\right) + c\left(\frac{n}{b^2}\right)^k\right) + c\left(\frac{n}{b}\right)^k\right) + cn^k \\ &\dots \\ &= a\left(a\left(\dots\left(aT\left(\frac{n}{b^m}\right) + c\left(\frac{n}{b^{m-1}}\right)^k\right) + \dots\right) + c\left(\frac{n}{b}\right)^k\right) + cn^k \end{aligned}$$

Assuming  $T(1) = c$ ,

$$T(n) = c \sum_{i=0}^m a^{m-i} b^{ik} = ca^m \sum_{i=0}^m \left(\frac{b^k}{a}\right)^i.$$

Three cases:  $\frac{b^k}{a} < 1$ ,  $\frac{b^k}{a} = 1$ , and  $\frac{b^k}{a} > 1$ .

# Divide and Conquer Relations (cont.)

## Theorem (3.4)

The solution of the recurrence relation  $T(n) = aT(n/b) + cn^k$ , where  $a$  and  $b$  are integer constants,  $a \geq 1$ ,  $b \geq 2$ , and  $c$  and  $k$  are positive constants, is

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > b^k \\ O(n^k \log n) & \text{if } a = b^k \\ O(n^k) & \text{if } a < b^k \end{cases}$$

# Useful Facts

## Harmonic series

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + O(1/n),$$

where  $\gamma = 0.577\dots$  is Euler's constant.

## Sum of logarithms

$$\begin{aligned} \sum_{i=1}^n \lfloor \log_2 i \rfloor &= (n+1) \lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2 \\ &= \Theta(n \log n). \end{aligned}$$



## Useful Facts (cont.)

- 🌐 Bounding a summation by an integral:  
If  $f(x)$  is monotonically *increasing*, then

$$\sum_{i=1}^n f(i) \leq \int_1^{n+1} f(x) dx.$$

If  $f(x)$  is monotonically *decreasing*, then

$$\sum_{i=1}^n f(i) \leq f(1) + \int_1^n f(x) dx.$$

- 🌐 Stirling's approximation

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(1/n)).$$