Algorithms 2017: String Processing

(Based on [Manber 1989])

Yih-Kuen Tsay

1 Data Compression

Data Compression

Problem 1. *Given a text (a sequence of characters), find an encoding for the characters that satisfies the prefix constraint and that minimizes the total number of bits needed to encode the text.*

The *prefix constraint* states that the prefixes of an encoding of one character must not be equal to a complete encoding of another character.

Denote the characters by c_1, c_2, $\cdots$, c_n and their frequencies by f_1, f_2, $\cdots$, f_n. Given an encoding E in which a bit string s_i represents c_i, the length (number of bits) of the text encoded by using E is $\sum_{i=1}^{n} |s_i| \cdot f_i$.

A Code Tree

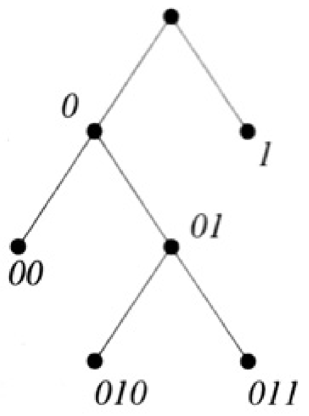


Figure 6.17 The tree representation of encoding.

Source: [Manber 1989].

A Huffman Tree

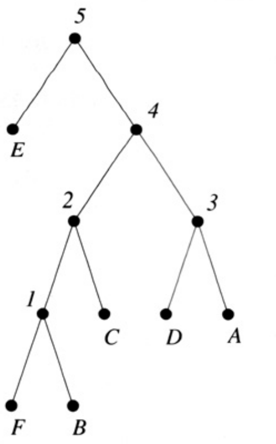


Figure 6.19 The Huffman tree for example 6.1.

Source: [Manber 1989]. (Frequencies: A: 5, B: 2, C: 3, D: 4, E: 10, F:1)

Huffman Encoding

```
Algorithm Huffman_Encoding ($S, f$);
    insert all characters into a heap $H$
        according to their frequencies;
    while $H$ not empty do
        if $H$ contains only one character $X$ then
            make $X$ the root of $T$
        else
            delete $X$ and $Y$ with lowest frequencies;
                from $H$;
            create $Z$ with a frequency equal to the
                sum of the frequencies of $X$ and $Y$;
            insert $Z$ into $H$;
            make $X$ and $Y$ children of $Z$ in $T$
```

What is its time complexity?

2 String Matching

String Matching

Problem 2. *Given two strings* A *(*$= a_1 a_2 \cdots a_n$*) and* B *(*$= b_1 b_2 \cdots b_m$*), find the first occurrence (if any) of* B *in* A*. In other words, find the smallest* k *such that, for all* i*,* $1 \leq i \leq m$*, we have* $a_{k-1+i} = b_i$*.*

A (non-empty) *substring* of a string A is a consecutive sequence of characters $a_i a_{i+1} \cdots a_j$ $(i \leq j)$ from A.

Straightforward String Matching

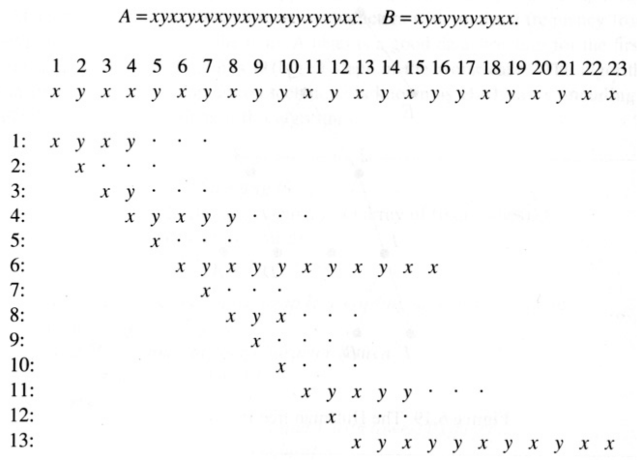


Figure 6.20 An example of a straightforward string matching.

Source: [Manber 1989].

Straightforward String Matching (cont.)

- What is the time complexity?
  - B ($= b_1 b_2 \cdots b_m$) may be compared against
    - $a_1 a_2 \cdots a_m$,
    - $a_2 a_3 \cdots a_{m+1}$,
    - $\ldots$, and
    - $a_{n-m+1} a_{n-m+2} \cdots a_n$
  - For example, $A = xxxx \ldots xxxy$ and $B = xxxy$.
- So, the time complexity is $O(m \times n)$.
- We will exam the cause of defficiency.
- We then study an efficient algorithm, which is linear-time with a preprocessing stage.

Matching Against Itself

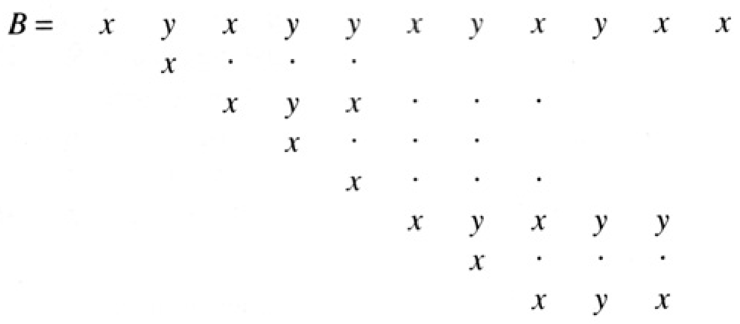
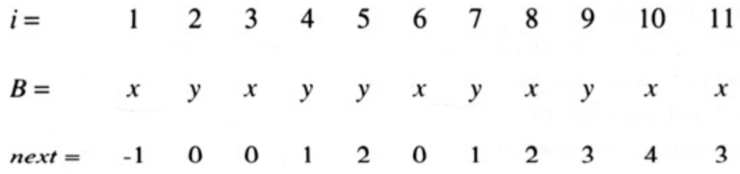


Figure 6.21 Matching the pattern against itself.

Source: [Manber 1989].

The Values of *next*



$i =$	1	2	3	4	5	6	7	8	9	10	11
$B =$	x	y	x	y	y	x	y	x	y	x	x
$next =$	-1	0	0	1	2	0	1	2	3	4	3

Figure 6.22 The values of *next*.

Source: [Manber 1989].

The value of $next[j]$ tells the length of the longest proper prefix that is equal to a suffix of $b_1 b_2 \ldots b_{j-1}$.

The KMP Algorithm

```
Algorithm String_Match $(A, n, B, m)$;
begin
    $j := 1$;  $i := 1$;
    $Start := 0$;
    while $Start = 0$ and $i \leq n$ do
        if $B[j] = A[i]$ then
            $j := j + 1$;  $i := i + 1$
        else
            $j := next[j] + 1$;
            if $j = 0$ then
                $j := 1$;  $i := i + 1$;
        if $j = m + 1$ then $Start := i - m$
end
```

The KMP Algorithm (cont.)

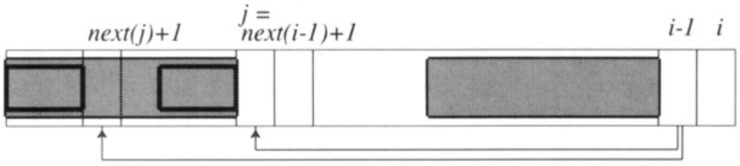


Figure 6.24 Computing next(i).

Source: [Manber 1989].

The KMP Algorithm (cont.)

```
Algorithm Compute_Next $(B, m)$;
begin
    $next[1] := -1$;  $next[2] := 0$;
    for $i := 3$ to $m$ do
        $j := next[i-1] + 1$;
        while $B[i-1] \neq B[j]$ and $j > 0$ do
            $j := next[j] + 1$;
        $next[i] := j$
end
```

The KMP Algorithm (cont.)

- What is its time complexity?
  - Because of backtracking, a_i may be compared against
    - b_j,
    - b_{j-1},
    - …, and
    - b_2
  - However, for these to happen, each of $a_{i-j+2}, a_{i-j+3}, \ldots, a_{i-1}$ was compared against the corresponding character in $b_1 b_2 \ldots b_{j-1}$ just once.
  - We may re-assign the costs of comparing a_i against $b_{j-1}, b_{j-2}, \ldots, b_2$ to those of comparing $a_{i-j+2} a_{i-j+3} \ldots a_{i-1}$ against $b_1 b_2 \ldots b_{j-1}$.
- Every a_i is incurred the cost of at most two comparisons.
- So, the time complexity is $O(n)$.

3 String Editing

String Editing

Problem 3. *Given two strings A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the minimum number of changes required to change A character by character such that it becomes equal to B.*

Three types of changes (or edit steps) allowed: (1) insert, (2) delete, and (3) replace.

String Editing (cont.)

Let $C(i, j)$ denote the minimum cost of changing $A(i)$ to $B(j)$, where $A(i) = a_1 a_2 \cdots a_i$ and $B(j) = b_1 b_2 \cdots b_j$.

$$C(i,j) = \min \begin{cases} C(i-1,j)+1 & \text{(deleting } a_i\text{)} \\ C(i,j-1)+1 & \text{(inserting } b_j\text{)} \\ C(i-1,j-1)+1 & (a_i \rightarrow b_j) \\ C(i-1,j-1) & (a_i = b_j) \end{cases}$$

String Editing (cont.)

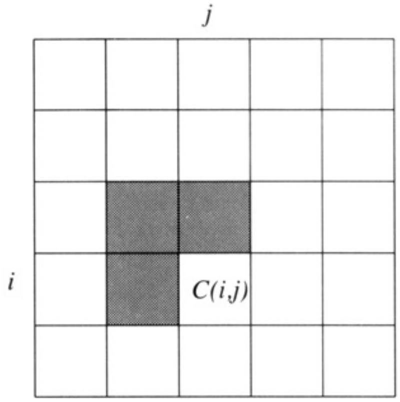


Figure 6.26 The dependencies of $C(i, j)$.

Source: [Manber 1989].

String Editing (cont.)

```
Algorithm Minimum_Edit_Distance $(A, n, B, m)$;
    for $i := 0$ to $n$ do $C[i, 0] := i$;
    for $j := 1$ to $m$ do $C[0, j] := j$;
    for $i := 1$ to $n$ do
        for $j := 1$ to $m$ do
            $x := C[i-1, j] + 1$;
            $y := C[i, j-1] + 1$;
            if $a_i = b_j$ then
                $z := C[i-1, j-1]$
            else
                $z := C[i-1, j-1] + 1$;
        $C[i, j] := min(x, y, z)$
```