

Searching and Sorting

(Based on [Manber 1989])

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Searching a Sorted Sequence

Problem

Let x_1, x_2, \dots, x_n be a sequence of real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$. Given a real number z , we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.

Searching a Sorted Sequence

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Idea: cut the search space in half by asking only one question.

Binary Search

```
function Find (z, Left, Right) : integer;  
begin  
  if Left = Right then  
    if  $X[Left] = z$  then Find := Left  
    else Find := 0  
  else  
    Middle :=  $\lceil \frac{Left+Right}{2} \rceil$ ;  
    if  $z < X[Middle]$  then  
      Find := Find(z, Left, Middle - 1)  
    else  
      Find := Find(z, Middle, Right)  
end
```

Binary Search (cont.)


```
Algorithm Binary_Search ( $X, n, z$ );  
begin  
     $Position := Find(z, 1, n)$ ;  
end
```

Searching a Cyclically Sorted Sequence

Problem


Given a *cyclically sorted* list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

Example 1:

 1 2 3 4 5 6 7 8
[5 6 7 0 1 2 3 4]

 The 4th is the minimal element.

Example 2:

 1 2 3 4 5 6 7 8
[0 1 2 3 4 5 6 7]

 The 1st is the minimal element.

Searching a Cyclically Sorted Sequence

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Given a *cyclically sorted* list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

🌐 Example 1:

☀️ 1 2 3 4 5 6 7 8
 [5 6 7 0 1 2 3 4]

☀️ The 4th is the minimal element.

🌐 Example 2:

☀️ 1 2 3 4 5 6 7 8
 [0 1 2 3 4 5 6 7]

☀️ The 1st is the minimal element.

🌐 To cut the search space in half, what question should we ask?

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);

begin

Position := *Cyclic_Find*(1, n);

end

function Cyclic_Find (*Left*, *Right*) : *integer*;

begin

if *Left* = *Right* **then** *Cyclic_Find* := *Left*

else

Middle := $\lfloor \frac{Left+Right}{2} \rfloor$;

if $X[Middle] < X[Right]$ **then**

Cyclic_Find := *Cyclic_Find*(*Left*, *Middle*)

else


Cyclic_Find := *Cyclic_Find*(*Middle* + 1, *Right*)


end


“Fixpoints”


Problem


Given a sorted sequence of *distinct* integers a_1, a_2, \dots, a_n , determine whether there exists an index i such that $a_i = i$.


 Example 1:


$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$

 $a_4 = 4$ (there are more ...).

 Example 2:


$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$


 There is no i such that $a_i = i$.

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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$

 $a_4 = 4$ (there are more ...).

 Example 2:


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 There is no i such that $a_i = i$.

 Again, can we cut the search space in half by asking only one question?

A Special Binary Search

```
function Special_Find (Left, Right) : integer;  
begin  
  if Left = Right then  
    if A[Left] = Left then Special_Find := Left  
    else Special_Find := 0  
  else  
    Middle :=  $\lfloor \frac{\textit{Left} + \textit{Right}}{2} \rfloor$ ;  
    if A[Middle] < Middle then  
      Special_Find := Special_Find(Middle + 1, Right)  
    else  
      Special_Find := Special_Find(Left, Middle)  
  end
```

A Special Binary Search (cont.)

```
Algorithm Special_Binary_Search ( $A, n$ );  
begin  
     $Position := Special\_Find(1, n)$ ;  
end
```

Stuttering Subsequence

Problem

Given two sequences A and B , find the maximal value of i such that B^i is a subsequence of A .

- 🌐 If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxxyyzzzzzzxxx$, etc.
- 🌐 B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- 🌐 For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.

Interpolation Search

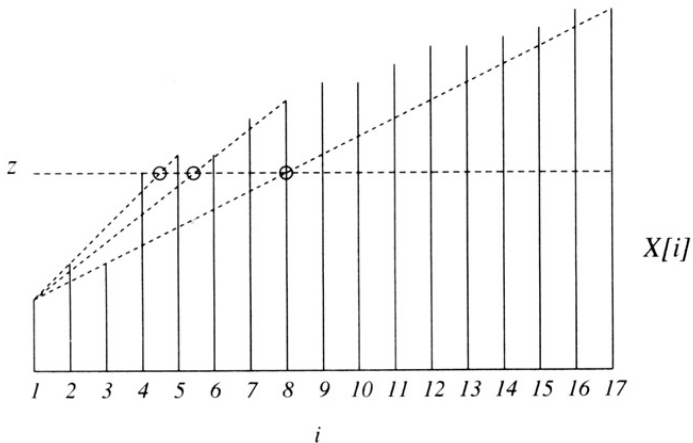


Figure 6.4 Interpolation search.

Source: [Manber 1989].

Interpolation Search (cont.)

```
function Int_Find (z, Left, Right) : integer;  
begin  
  if  $X[Left] = z$  then Int_Find := Left  
  else if Left = Right or  $X[Left] = X[Right]$  then  
    Int_Find := 0  
  else  
     $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$ ;  
    if  $z < X[Next\_Guess]$  then  
      Int_Find := Int_Find(z, Left, Next_Guess - 1)  
    else  
      Int_Find := Int_Find(z, Next_Guess, Right)  
  end
```

Interpolation Search (cont.)


```
Algorithm Interpolation_Search ( $X, n, z$ );  
begin  
    if  $z < X[1]$  or  $z > X[n]$  then  $Position := 0$   
    else  $Position := Int\_Find(z, 1, n)$ ;  
end
```


Problem

Given n numbers x_1, x_2, \dots, x_n , arrange them in increasing order. In other words, find a sequence of distinct indices $1 \leq i_1, i_2, \dots, i_n \leq n$, such that $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$.

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

Using Balanced Search Trees

-  Balanced search trees, such as AVL trees, may be used for sorting:
1. Create an empty tree.
 2. Insert the numbers one by one to the tree.
 3. Traverse the tree and output the numbers.

Using Balanced Search Trees

- 🌐 Balanced search trees, such as AVL trees, may be used for sorting:
 1. Create an empty tree.
 2. Insert the numbers one by one to the tree.
 3. Traverse the tree and output the numbers.
- 🌐 What's the time complexity? Suppose we use an AVL tree.

Radix Sort

```
Algorithm Straight_Radix ( $X, n, k$ );  
begin  
    put all elements of  $X$  in a queue  $GQ$ ;  
    for  $i := 1$  to  $d$  do  
        initialize queue  $Q[i]$  to be empty  
    for  $i := k$  downto  $1$  do  
        while  $GQ$  is not empty do  
            pop  $x$  from  $GQ$ ;  
             $d :=$  the  $i$ -th digit of  $x$ ;  
            insert  $x$  into  $Q[d]$ ;  
        for  $t := 1$  to  $d$  do  
            insert  $Q[t]$  into  $GQ$ ;  
    for  $i := 1$  to  $n$  do  
        pop  $X[i]$  from  $GQ$   
end
```

Merge Sort

Algorithm Mergesort (X, n);
begin $M_Sort(1, n)$ **end**

procedure M_Sort ($Left, Right$);
begin

if $Right - Left = 1$ **then**

if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$

else if $Left \neq Right$ **then**

$Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;

$M_Sort(Left, Middle - 1)$;

$M_Sort(Middle, Right)$;

Merge Sort (cont.)

```
i := Left; j := Middle; k := 0;
while (i ≤ Middle − 1) and (j ≤ Right) do
    k := k + 1;
    if X[i] ≤ X[j] then
        TEMP[k] := X[i]; i := i + 1
    else TEMP[k] := X[j]; j := j + 1;
if j > Right then
    for t := 0 to Middle − 1 − i do
        X[Right − t] := X[Middle − 1 − t]
    for t := 0 to k − 1 do
        X[Left + t] := TEMP[t]
end
```

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
②	⑥	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	⑤	⑧	10	9	12	1	15	7	3	13	4	11	16	14
②	⑤	⑥	⑧	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	⑨	⑩	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	①	⑫	15	7	3	13	4	11	16	14
2	5	6	8	①	⑨	⑩	⑫	15	7	3	13	4	11	16	14
①	②	⑤	⑥	⑧	⑨	⑩	⑫	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	⑦	⑮	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	③	⑬	4	11	16	14
1	2	5	6	8	9	10	12	③	⑦	⑬	⑮	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	⑭	⑰
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	⑭	⑰
1	2	5	6	8	9	10	12	③	④	⑦	⑪	⑬	⑭	⑮	⑰
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑰

Figure 6.8 An example of mergesort. The first row is in the initial order. Each row illustrates either an exchange operation or a merge. The numbers that are involved in the current operation are circled.

Source: [Manber 1989].

Quick Sort

```
Algorithm Quicksort ( $X, n$ );  
begin  
     $Q\_Sort(1, n)$   
end  
  
procedure Q_Sort ( $Left, Right$ );  
begin  
    if  $Left < Right$  then  
         $Partition(X, Left, Right)$ ;  
         $Q\_Sort(Left, Middle - 1)$ ;  
         $Q\_Sort(Middle + 1, Right)$   
    end
```


Quick Sort (cont.)

Algorithm Partition ($X, Left, Right$);

begin

$pivot := X[left];$

$L := Left; R := Right;$

while $L < R$ **do**

while $X[L] \leq pivot$ and $L \leq Right$ **do** $L := L + 1;$

while $X[R] > pivot$ and $R \geq Left$ **do** $R := R - 1;$

if $L < R$ **then** $swap(X[L], X[R]);$

$Middle := R;$

$swap(X[Left], X[Middle])$

end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	4	5	10	9	12	1	15	7	3	13	8	11	16	14
6	2	4	5	3	9	12	1	15	7	10	13	8	11	16	14
6	2	4	5	3	1	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14

Figure 6.10 Partition of an array around the pivot 6.

Source: [Manber 1989].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure 6.12 An example of quicksort. The first line is the initial input. A new pivot is selected in each line. The pivots are circled. When a single number appears between two pivots it is obviously in the right position.

Source: [Manber 1989].

Average-Case Complexity of Quick Sort

🌐 When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i - 1) + T(n - i), \text{ where } n \geq 2.$$

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i - 1) + T(n - i), \text{ where } n \geq 2.$$

The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \frac{1}{n} \sum_{i=1}^n (T(i - 1) + T(n - i)) \\ &= n - 1 + \frac{1}{n} \sum_{i=1}^n T(i - 1) + \frac{1}{n} \sum_{i=1}^n T(n - i) \\ &= n - 1 + \frac{1}{n} \sum_{j=0}^{n-1} T(j) + \frac{1}{n} \sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \frac{2}{n} \sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history,
 $T(n) = O(n \log n)$.

Heap Sort

```
Algorithm Heapsort ( $A, n$ );  
begin  
    Build_Heap( $A$ );  
    for  $i := n$  downto 2 do  
        swap( $A[1], A[i]$ );  
        Rearrange_Heap( $i - 1$ )  
end
```

Heap Sort (cont.)

```
procedure Rearrange_Heap (k);  
begin  
    parent := 1;  
    child := 2;  
    while child ≤ k - 1 do  
        if A[child] < A[child + 1] then  
            child := child + 1;  
        if A[child] > A[parent] then  
            swap(A[parent], A[child]);  
            parent := child;  
            child := 2 * child  
        else child := k  
end
```

Heap Sort (cont.)

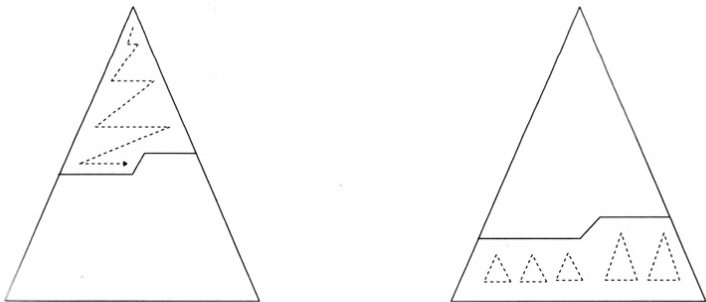


Figure 6.14 Top down and bottom up heap construction.

Source: [Manber 1989].

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	14	15	7	3	13	4	11	16	1
2	6	8	5	10	9	16	14	15	7	3	13	4	11	12	1
2	6	8	5	10	13	16	14	15	7	3	9	4	11	12	1
2	6	8	15	10	13	16	14	5	7	3	9	4	11	12	1
2	6	16	15	10	13	12	14	5	7	3	9	4	11	8	1
2	15	16	14	10	13	12	6	5	7	3	9	4	11	8	1
16	15	13	14	10	9	12	6	5	7	3	2	4	11	8	1

Figure 6.15 An example of building a heap bottom up. The numbers on top are the indices. The circled numbers are those that have been exchanged on that step.

Source: [Manber 1989] (6 and 2 in the first row should be swapped).

A Lower Bound for Sorting

- 🌐 A **lower bound** for a particular problem is a proof that *no algorithm* can solve the problem better.
- 🌐 We typically define a **computation model** and consider only those algorithms that fit in the model.
- 🌐 **Decision trees** model computations performed by *comparison-based* algorithms.

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Theorem (Theorem 6.1)

Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.

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- 🌐 The obvious solution requires $(n - 1) + (n - 2) (= 2n - 3)$ comparisons between elements.
- 🌐 Can we do better? Which comparisons could have been avoided?

Order Statistics: K th-Smallest

Problem

Given a sequence $S = x_1, x_2, \dots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the k th-smallest element in S .

Order Statistics: K th-Smallest (cont.)

```
procedure Select (Left, Right, k);  
begin  
  if Left = Right then  
    Select := Left  
  else Partition(X, Left, Right);  
    let Middle be the output of Partition;  
    if Middle - Left + 1  $\geq$  k then  
      Select(Left, Middle, k)  
    else  
      Select(Middle + 1, Right, k - (Middle - Left + 1))  
end
```


Order Statistics: K th-Smallest (cont.)

The nested “if” statement may be simplified:

```
procedure Select (Left, Right, k);  
begin  
  if Left = Right then  
    Select := Left  
  else Partition(X, Left, Right);  
    let Middle be the output of Partition;  
    if Middle  $\geq$  k then  
      Select(Left, Middle, k)  
    else  
      Select(Middle + 1, Right, k)  
end
```

Order Statistics: K th-Smallest (cont.)

```
Algorithm Selection ( $X, n, k$ );  
begin  
    if ( $k < 1$ ) or ( $k > n$ ) then print "error"  
    else  $S := \text{Select}(1, n, k)$   
end
```

Finding a Majority

Problem

Given a sequence of numbers, find the majority in the sequence or determine that none exists.

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Finding a Majority (cont.)

```
Algorithm Majority ( $X, n$ );  
begin  
   $C := X[1]; M := 1;$   
  for  $i := 2$  to  $n$  do  
    if  $M = 0$  then  
       $C := X[i]; M := 1$   
    else  
      if  $C = X[i]$  then  $M := M + 1$   
      else  $M := M - 1;$ 
```

Finding a Majority (cont.)

```
if  $M = 0$  then Majority := -1  
else  
    Count := 0;  
    for  $i := 1$  to  $n$  do  
        if  $X[i] = C$  then Count := Count + 1;  
    if  $Count > n/2$  then Majority :=  $C$   
    else Majority := -1  
end
```