Algorithms 2018: Searching and Sorting

(Based on [Manber 1989])

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1 Binary Search

Searching a Sorted Sequence

Problem 1. *Let* $x_1, x_2, \cdots, x_n$ *be a sequence of real numbers such that* $x_1 \leq x_2 \leq \cdots \leq x_n$*. Given a real number* z*, we want to find whether* z *appears in the sequence, and, if it does, to find an index* i *such that* $x_i = z$*.*

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

```
function Find $(z, Left, Right) : integer$;
begin
    if $Left = Right$ then
       if $X[Left] = z$ then $Find := Left$
       else $Find := 0$
    else
       $Middle := \lceil \frac{Left+Right}{2} \rceil$;
       if $z < X[Middle]$ then
          $Find := Find(z, Left, Middle - 1)$
       else
          $Find := Find(z, Middle, Right)$
end
```

Binary Search (cont.)

```
Algorithm Binary_Search $(X, n, z)$;
begin
    $Position := Find(z, 1, n)$;
end
```

1.1 Cyclically Sorted Sequence

Searching a Cyclically Sorted Sequence

Problem 2. *Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).*

- Example 1:
  - $\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 &] \end{array}$
  - The 4th is the minimal element.
- Example 2:
  - $\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 &] \end{array}$
  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

  /* If $X[Middle] < X[Right]$, then the minimal is in the left half (including $X[Middle]$; otherwise, it is in the right half (excluding $X[Middle]$). */

Cyclic Binary Search

```
Algorithm Cyclic_Binary_Search (X, n);
begin
    Position := Cyclic_Find(1, n);
end

function Cyclic_Find (Left, Right) : integer;
begin
    if Left = Right then Cyclic_Find := Left
    else
        Middle := ⌊(Left+Right)/2⌋;
        if X[Middle] < X[Right] then
            Cyclic_Find := Cyclic_Find(Left, Middle)
        else
            Cyclic_Find := Cyclic_Find(Middle + 1, Right)
end
```

1.2 “Fixpoints”

“Fixpoints”

Problem 3. *Given a sorted sequence of distinct integers $a_1, a_2, \cdots, a_n$, determine whether there exists an index i such that $a_i = i$.*

- Example 1:
  - $\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$
  - $a_4 = 4$ (there are more ...).

- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$
  - There is no i such that $a_i = i$.
- Again, can we cut the search space in half by asking only one question?

  /* As the numbers are distinct, they increase or decrease at least as fast as the indices (which always increase or decrease by one). If $X[Middle] < Middle$, then the fixpoint (if it exists) must be in the left half (excluding $X[Middle]$; otherwise, it must be in the right half (including $X[Middle]$). */

A Special Binary Search

```
function Special_Find (Left, Right) : integer;
begin
    if $Left = Right$ then
        if $A[Left] = Left$ then $Special\_Find := Left$
        else $Special\_Find := 0$
    else
        $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
        if $A[Middle] < Middle$ then
            $Special\_Find := Special\_Find(Middle + 1, Right)$
        else
            $Special\_Find := Special\_Find(Left, Middle)$
end
```

A Special Binary Search (cont.)

```
Algorithm Special_Binary_Search (A, n);
begin
    $Position := Special\_Find(1, n)$;
end
```

1.3 Stuttering Subsequence

Stuttering Subsequence

Problem 4. *Given two sequences A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the maximal value of i such that B^i is a subsequence of A.*

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \le i \le j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.
  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.
  Time complexity: $O(n \log \frac{n}{m})$.

Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

/* Try 2^0, 2^1, 2^2, $\cdots$, 2^{k-1}, and 2^k sequentially. If the target falls between 2^{k-1} and 2^k, apply binary search within that region. */

2 Interpolation Search

Interpolation Search

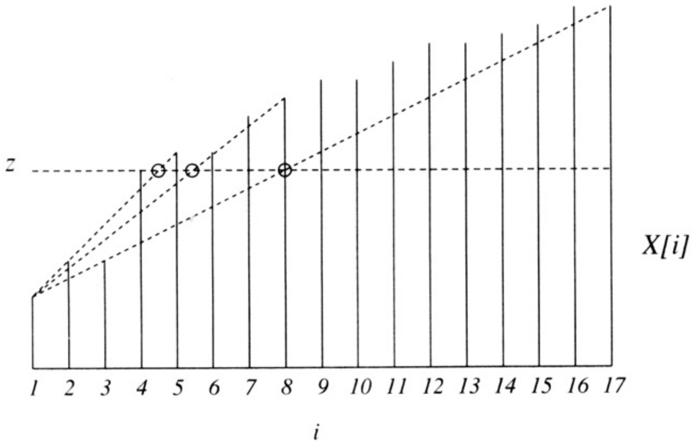
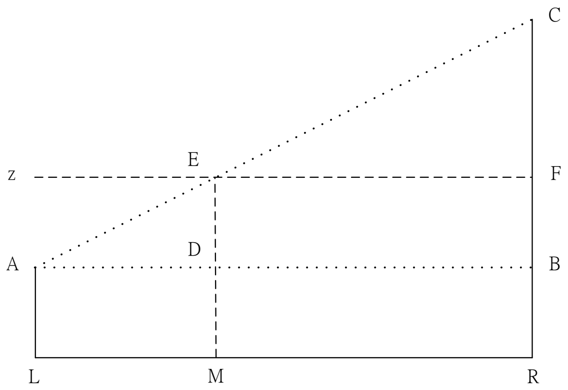


Figure 6.4 Interpolation search.

Source: [Manber 1989].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

```
function Int_Find $(z, Left, Right) : integer$;
begin
    if $X[Left] = z$ then $Int\_Find := Left$
    else if $Left = Right$ or $X[Left] = X[Right]$ then
        $Int\_Find := 0$
    else
        $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$;
        if $z < X[Next\_Guess]$ then
            $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
        else
            $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end
```

/* $Next\_Guess - Left = |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}| \approx \lceil \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$ */

Interpolation Search (cont.)

```
Algorithm Interpolation_Search $(X, n, z)$;
begin
    if $z < X[1]$ or $z > X[n]$ then $Position := 0$
    else $Position := Int\_Find(z, 1, n)$;
end
```

3 Sorting

Sorting

Problem 5. *Given n numbers $x_1, x_2, \cdots, x_n$, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \le i_1, i_2, \cdots, i_n \le n$, such that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$.*

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

3.1 Using Balanced Search Trees

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
    1. Create an empty tree.
    2. Insert the numbers one by one to the tree.
    3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

3.2 Radix Sort

Radix Sort

```
Algorithm Straight_Radix $(X, n, k)$;
begin
    put all elements of $X$ in a queue $GQ$;
    for $i := 1$ to $d$ do
        initialize queue $Q[i]$ to be empty
    for $i := k$ downto $1$ do
        while $GQ$ is not empty do
                pop $x$ from $GQ$;
                $d :=$ the $i$-th digit of $x$;
                insert $x$ into $Q[d]$;
        for $t := 1$ to $d$ do
            insert $Q[t]$ into $GQ$;
    for $i := 1$ to $n$ do
        pop $X[i]$ from $GQ$
end
```

Time complexity: $O(nk)$.

3.3 Merge Sort

Merge Sort

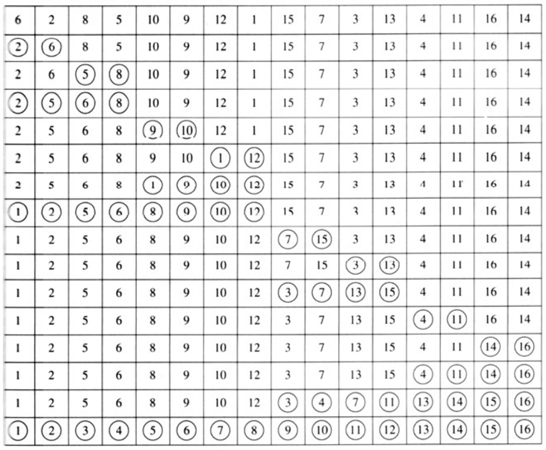
```
Algorithm Mergesort $(X, n)$;
begin  $M\_Sort(1, n)$  end

procedure M_Sort $(Left, Right)$;
begin
    if $Right - Left = 1$ then
        if $X[Left] > X[Right]$ then $swap(X[Left], X[Right])$
    else if $Left \neq Right$ then
            $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
            $M\_Sort(Left, Middle - 1)$;
            $M\_Sort(Middle, Right)$;
```

Merge Sort (cont.)

```
            $i := Left$;  $j := Middle$;  $k := 0$;
            while $(i \leq Middle - 1)$ and $(j \leq Right)$ do
                $k := k + 1$;
                if $X[i] \leq X[j]$ then
                    $TEMP[k] := X[i]$;  $i := i + 1$
                else $TEMP[k] := X[j]$;  $j := j + 1$;
            if $j > Right$ then
                for $t := 0$ to $Middle - 1 - i$ do
                    $X[Right - t] := X[Middle - 1 - t]$
            for $t := 0$ to $k - 1$ do
                $X[Left + t] := TEMP[1 + t]$
end
```

Time complexity: $O(n \log n)$.

Merge Sort (cont.)



6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
②	⑥	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	⑤	⑧	10	9	12	1	15	7	3	13	4	11	16	14
②	⑤	⑥	⑧	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	⑨	⑩	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	①	⑫	15	7	3	13	4	11	16	14
2	5	6	8	①	⑨	⑩	⑫	15	7	3	13	4	11	16	14
①	②	⑤	⑥	⑧	⑨	⑩	⑫	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	⑦	⑮	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	③	⑬	4	11	16	14
1	2	5	6	8	9	10	12	③	⑦	⑬	⑮	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	⑭	⑯
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	⑭	⑯
1	2	5	6	8	9	10	12	③	④	⑦	⑪	⑬	⑭	⑮	⑯
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯

Figure 6.8 An example of mergesort. The first row is in the initial order. Each row illustrates either an exchange operation or a merge. The numbers that are involved in the current operation are circled.

Source: [Manber 1989].

3.4 Quick Sort

Quick Sort

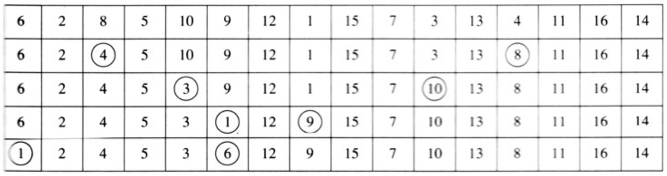
```
Algorithm Quicksort $(X, n)$;
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort $(Left, Right)$;
begin
    if $Left < Right$ then
        $Partition(X, Left, Right)$;
        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end
```

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

Quick Sort (cont.)

```
Algorithm Partition $(X, Left, Right)$;
begin
    $pivot := X[left]$;
    $L := Left$;  $R := Right$;
    while $L < R$ do
            while $X[L] \leq pivot$ and $L \leq Right$ do $L := L + 1$;
            while $X[R] > pivot$ and $R \geq Left$ do $R := R - 1$;
            if $L < R$ then $swap(X[L], X[R])$;
    $Middle := R$;
    $swap(X[Left], X[Middle])$
end
```

Quick Sort (cont.)

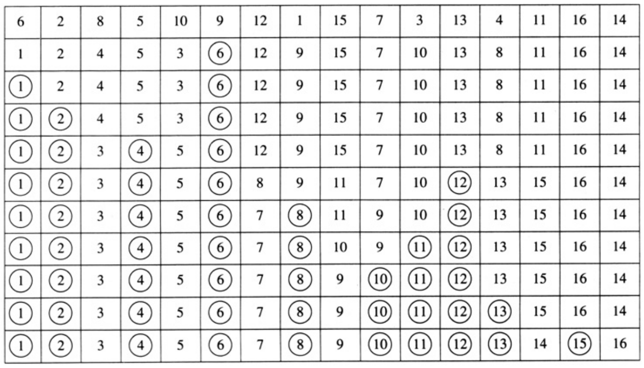


6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	④	5	10	9	12	1	15	7	3	13	⑧	11	16	14
6	2	4	5	③	9	12	1	15	7	⑩	13	8	11	16	14
6	2	4	5	3	①	12	⑨	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14

Figure 6.10 Partition of an array around the pivot 6.

Source: [Manber 1989].

Quick Sort (cont.)



6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure 6.12 An example of quicksort. The first line is the initial input. A new pivot is selected in each line. The pivots are circled. When a single number appears between two pivots it is obviously in the right position.

Source: [Manber 1989].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

  $$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

  The average running time will then be

  $$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\textstyle\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

3.5 Heap Sort

Heap Sort

```
Algorithm Heapsort $(A, n)$;
begin
    $Build\_Heap(A)$;
    for $i := n$ downto $2$ do
        $swap(A[1], A[i])$;
        $Rearrange\_Heap(i - 1)$
end
```

Time complexity: $O(n \log n)$

Heap Sort (cont.)

```
procedure Rearrange_Heap $(k)$;
begin
    $parent := 1$;
    $child := 2$;
    while $child \leq k - 1$ do
            if $A[child] < A[child + 1]$ then
                $child := child + 1$;
            if $A[child] > A[parent]$ then
                $swap(A[parent], A[child])$;
                $parent := child$;
                $child := 2 * child$
            else $child := k$
end
```

Heap Sort (cont.)

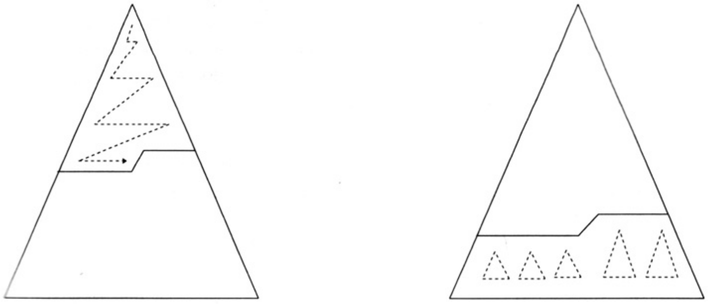


Figure 6.14 Top down and bottom up heap construction.

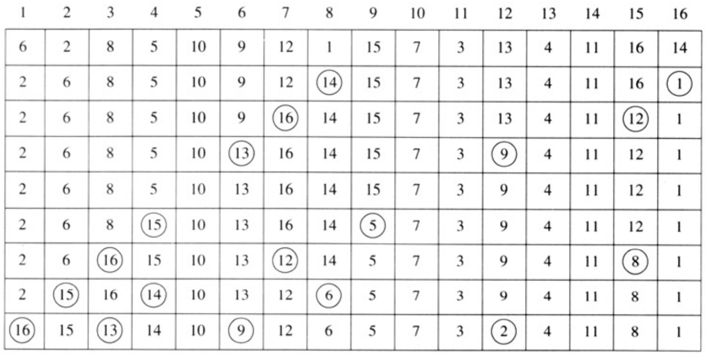
Source: [Manber 1989].

How do the two approaches compare?

/* Top down: $O(n \log n)$.

Bottom up: $O(\text{sum of the heights of all nodes}) = O(n)$. Consider a full binary tree of height h. From an excercise problem in HW#2, we know that "sum of the heights of all nodes" of the tree equals $2^{h+1} - (h+2) \leq 2^{h+1} - 1 = n$. */

Building a Heap Bottom Up



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	⑭	15	7	3	13	4	11	16	①
2	6	8	5	10	9	⑯	14	15	7	3	13	4	11	⑫	1
2	6	8	5	10	⑬	16	14	15	7	3	⑨	4	11	12	1
2	6	8	5	10	13	16	14	15	7	3	9	4	11	12	1
2	6	8	⑮	10	13	16	14	⑤	7	3	9	4	11	12	1
2	6	⑯	15	10	13	⑫	14	5	7	3	9	4	11	⑧	1
2	⑮	16	⑭	10	13	12	⑥	5	7	3	9	4	11	8	1
⑯	15	⑬	14	10	⑨	12	6	5	7	3	②	4	11	8	1

Figure 6.15 An example of building a heap bottom up. The numbers on top are the indices. The circled numbers are those that have been exchanged on that step.

Source: [Manber 1989] (6 and 2 in the first row should be swapped).

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem 6 (Theorem 6.1)**.** *Every decision-tree algorithm for sorting has height* $\Omega(n \log n)$.

Proof idea: there must be at least $n!$ leaves, one for each possible outcome.

/* Recall Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$. The height of the decision tree must be at least $\log(n!)$, i.e., $\Omega(n \log n)$. */

Is the lower bound contradictory to the time complexity of radix sort?

4 Order Statistics

Order Statistics: Minimum and Maximum

Problem 7. *Find the maximum and minimum elements in a given sequence.*

- The obvious solution requires $(n-1) + (n-2)$ $(= 2n-3)$ comparisons between elements.
- Can we do better? Which comparisons could have been avoided?

Order Statistics: Kth-Smallest

Problem 8. *Given a sequence* $S = x_1, x_2, \cdots, x_n$ *of elements, and an integer* k *such that* $1 \leq k \leq n$*, find the* k*th-smallest element in* S*.*

Order Statistics: Kth-Smallest (cont.)

```
procedure Select ($Left, Right, k$);
begin
    if $Left = Right$ then
        $Select := Left$
    else $Partition(X, Left, Right)$;
        let $Middle$ be the output of $Partition$;
        if $Middle - Left + 1 \geq k$ then
            $Select(Left, Middle, k)$
        else
            $Select(Middle + 1, Right, k - (Middle - Left + 1))$
end
```

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

```
procedure Select ($Left, Right, k$);
begin
    if $Left = Right$ then
        $Select := Left$
    else $Partition(X, Left, Right)$;
        let $Middle$ be the output of $Partition$;
        if $Middle \geq k$ then
            $Select(Left, Middle, k)$
        else
            $Select(Middle + 1, Right, k)$
end
```

Order Statistics: Kth-Smallest (cont.)

```
Algorithm Selection ($X, n, k$);
begin
    if ($k < 1$) or ($k > n$) then print "error"
    else $S := Select(1, n, k)$
end
```

5 Finding a Majority

Finding a Majority

Problem 9. *Given a sequence of numbers, find the majority in the sequence or determine that none exists.*

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

/* If there is a majority, it is also a majority of the other $n - 2$ numbers. */

What if they are equal?

Finding a Majority (cont.)

```
Algorithm Majority $(X, n)$;
begin
    $C := X[1]$;  $M := 1$;
    for $i := 2$ to $n$ do
        if $M = 0$ then
            $C := X[i]$;  $M := 1$
        else
            if $C = X[i]$ then $M := M + 1$
            else $M := M - 1$;
```

Finding a Majority (cont.)

```
    if $M = 0$ then $Majority := -1$
    else
        $Count := 0$;
        for $i := 1$ to $n$ do
            if $X[i] = C$ then $Count := Count + 1$;
        if $Count > n/2$ then $Majority := C$
        else $Majority := -1$
end
```