Algorithms 2018: Searching and Sorting

(Based on [Manber 1989])

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1 Binary Search

Searching a Sorted Sequence

Problem 1. *Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.*

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

function Find $(z, Left, Right) : integer$;
begin
    if $Left = Right$ **then**
        if $X[Left] = z$ **then** $Find := Left$
        else $Find := 0$
    else
        $Middle := \lceil \frac{Left+Right}{2} \rceil$;
        if $z < X[Middle]$ **then**
            $Find := Find(z, Left, Middle - 1)$
        else
            $Find := Find(z, Middle, Right)$
end

Binary Search (cont.)

Algorithm Binary_Search (X, n, z);
begin
    $Position := Find(z, 1, n)$;
end

1.1 Cyclically Sorted Sequence

Searching a Cyclically Sorted Sequence

Problem 2. *Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).*

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 &] \end{array}$$
  - The 4th is the minimal element.
- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 &] \end{array}$$
  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

  /* If $X[Middle] < X[Right]$, then the minimal is in the left half (including $X[Middle]$; otherwise, it is in the right half (excluding $X[Middle]$). */

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);
begin
  $Position := Cyclic\_Find(1, n)$;
end

function Cyclic_Find $(Left, Right) : integer$;
begin
  if $Left = Right$ **then** $Cyclic\_Find := Left$
  else
    $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
    if $X[Middle] < X[Right]$ **then**
      $Cyclic\_Find := Cyclic\_Find(Left, Middle)$
    else
      $Cyclic\_Find := Cyclic\_Find(Middle + 1, Right)$
end

1.2 "Fixpoints"

"Fixpoints"

Problem 3. *Given a sorted sequence of distinct integers $a_1, a_2, \cdots, a_n$, determine whether there exists an index i such that $a_i = i$.*

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$
  - $a_4 = 4$ (there are more ...).

- Example 2:

  – $$\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccccccc} -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 \end{array} \right] \end{array}$$

  – There is no i such that $a_i = i$.

- Again, can we cut the search space in half by asking only one question?

  /* As the numbers are distinct, they increase or decrease at least as fast as the indices (which always increase or decrease by one). If $X[Middle] < Middle$, then the fixpoint (if it exists) must be in the left half (excluding $X[Middle]$; otherwise, it must be in the right half (including $X[Middle]$). */

A Special Binary Search

```
function Special_Find (Left, Right) : integer;
begin
    if Left = Right then
      if A[Left] = Left then Special_Find := Left
      else Special_Find := 0
    else
        Middle := ⌊(Left+Right)/2⌋;
        if A[Middle] < Middle then
          Special_Find := Special_Find(Middle + 1, Right)
        else
          Special_Find := Special_Find(Left, Middle)
end
```

A Special Binary Search (cont.)

```
Algorithm Special_Binary_Search (A, n);
begin
    Position := Special_Find(1, n);
end
```

1.3 Stuttering Subsequence

Stuttering Subsequence

Problem 4. *Given two sequences A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the maximal value of i such that B^i is a subsequence of A.*

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.

  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.

  Time complexity: $O(n \log \frac{n}{m})$.

Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

/* Try $2^0, 2^1, 2^2, \cdots, 2^{k-1}$, and 2^k sequentially. If the target falls between 2^{k-1} and 2^k, apply binary search within that region. */

2 Interpolation Search

Interpolation Search

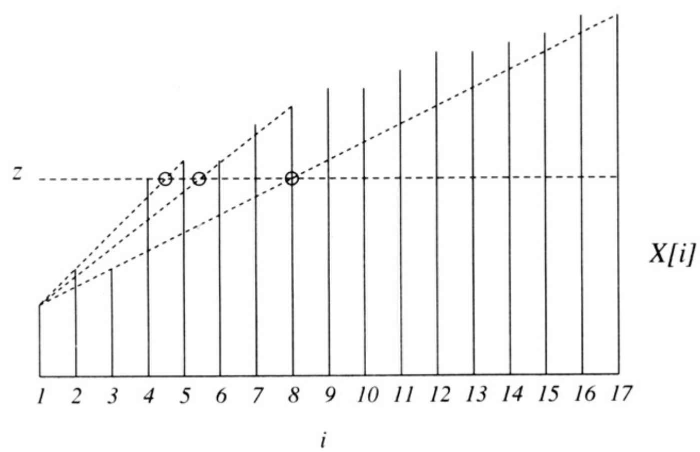
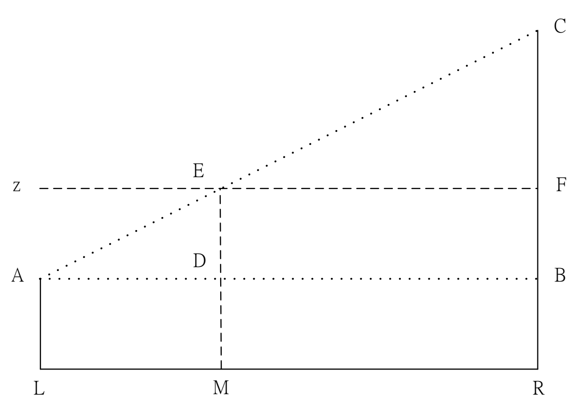


Figure 6.4 Interpolation search.

Source: [Manber 1989].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

function Int_Find $(z, Left, Right) : integer$;
begin
    if $X[Left] = z$ **then** $Int\_Find := Left$
    else if $Left = Right$ or $X[Left] = X[Right]$ **then**
        $Int\_Find := 0$
    else
        $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$;
        if $z < X[Next\_Guess]$ **then**
            $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
        else
            $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

/* $Next\_Guess - Left = |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}| \approx \lceil \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$ */

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
    if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
    else $Position := Int\_Find(z, 1, n)$;
end

3 Sorting

Sorting

Problem 5. *Given n numbers x_1, x_2, $\cdots$, x_n, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \leq i_1, i_2, \cdots, i_n \leq n$, such that $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$.*

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

3.1 Using Balanced Search Trees

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
    1. Create an empty tree.
    2. Insert the numbers one by one to the tree.
    3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

3.2 Radix Sort

Radix Sort

Algorithm Straight_Radix (X, n, k);
begin
    put all elements of X *in a queue* GQ;
    for $i := 1$ **to** d **do**
        initialize queue $Q[i]$ *to be empty*
    for $i := k$ **downto** 1 **do**
        while GQ *is not empty* **do**
                pop x *from* GQ;
                $d :=$ *the* i*-th digit of* x;
                insert x *into* $Q[d]$;
        for $t := 1$ **to** d **do**
            insert $Q[t]$ *into* GQ;
    for $i := 1$ **to** n **do**
        pop $X[i]$ *from* GQ
end

Time complexity: $O(nk)$.

3.3 Merge Sort

Merge Sort

Algorithm Mergesort (X, n);
begin $M\_Sort(1, n)$ **end**

procedure M_Sort $(Left, Right)$;
begin
    if $Right - Left = 1$ **then**
        if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$
    else if $Left \neq Right$ **then**
            $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
            $M\_Sort(Left, Middle - 1)$;
            $M\_Sort(Middle, Right)$;

Merge Sort (cont.)

            $i := Left$; $j := Middle$; $k := 0$;
            while $(i \leq Middle - 1)$ and $(j \leq Right)$ **do**
                    $k := k + 1$;
                    if $X[i] \leq X[j]$ **then**
                        $TEMP[k] := X[i]$; $i := i + 1$
                    else $TEMP[k] := X[j]$; $j := j + 1$;
            if $j > Right$ **then**
                for $t := 0$ **to** $Middle - 1 - i$ **do**
                    $X[Right - t] := X[Middle - 1 - t]$
            for $t := 0$ **to** $k - 1$ **do**
                $X[Left + t] := TEMP[t]$
end

Time complexity: $O(n \log n)$.

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	5	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	1	12	15	7	3	13	4	11	16	14
2	5	6	8	1	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	4	7	11	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Figure 6.8 An example of mergesort. The first row is in the initial order. Each row illustrates either an exchange operation or a merge. The numbers that are involved in the current operation are circled.

Source: [Manber 1989].

3.4 Quick Sort

Quick Sort

Algorithm Quicksort (X, n);
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort $(Left, Right)$;
begin
    if $Left < Right$ **then**
        $Partition(X, Left, Right)$;
        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

Quick Sort (cont.)

Algorithm Partition $(X, Left, Right)$;
begin
    $pivot := X[left]$;
    $L := Left$; $R := Right$;
    while $L < R$ **do**
        while $X[L] \leq pivot$ and $L \leq Right$ **do** $L := L + 1$;
        while $X[R] > pivot$ and $R \geq Left$ **do** $R := R - 1$;
        if $L < R$ **then** $swap(X[L], X[R])$;
    $Middle := R$;
    $swap(X[Left], X[Middle])$
end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	4	5	10	9	12	1	15	7	3	13	8	11	16	14
6	2	4	5	3	9	12	1	15	7	10	13	8	11	16	14
6	2	4	5	3	1	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14

Figure 6.10 Partition of an array around the pivot 6.

Source: [Manber 1989].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	3	4	5	6	12	9	15	7	10	13	8	11	16	14
1	2	3	4	5	6	8	9	11	7	10	12	13	15	16	14
1	2	3	4	5	6	7	8	11	9	10	12	13	15	16	14
1	2	3	4	5	6	7	8	10	9	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Figure 6.12 An example of quicksort. The first line is the initial input. A new pivot is selected in each line. The pivots are circled. When a single number appears between two pivots it is obviously in the right position.

Source: [Manber 1989].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\textstyle\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\textstyle\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

3.5 Heap Sort

Heap Sort

Algorithm Heapsort (A, n);
begin
  $Build\_Heap(A)$;
  for $i := n$ **downto** 2 **do**
    $swap(A[1], A[i])$;
    $Rearrange\_Heap(i-1)$
end

Time complexity: $O(n \log n)$

Heap Sort (cont.)

procedure Rearrange_Heap (k);
begin
  $parent := 1$;
  $child := 2$;
  while $child \leq k-1$ **do**
    if $A[child] < A[child+1]$ **then**
      $child := child + 1$;
    if $A[child] > A[parent]$ **then**
      $swap(A[parent], A[child])$;
      $parent := child$;
      $child := 2 * child$
    else $child := k$
end

Heap Sort (cont.)

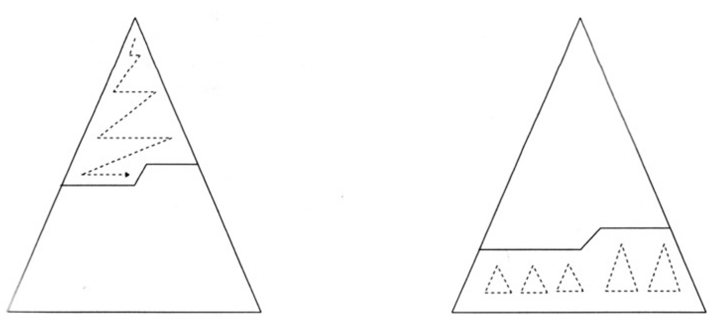


Figure 6.14 Top down and bottom up heap construction.

Source: [Manber 1989].

How do the two approaches compare?

/* Top down: $O(n \log n)$.

Bottom up: O(sum of the heights of all nodes) $= O(n)$. Consider a full binary tree of height h. From an excercise problem in HW#2, we know that "sum of the heights of all nodes" of the tree equals $2^{h+1} - (h+2) \leq 2^{h+1} - 1 = n$. */

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	14	15	7	3	13	4	11	16	1
2	6	8	5	10	9	16	14	15	7	3	13	4	11	12	1
2	6	8	5	10	13	16	14	15	7	3	9	4	11	12	1
2	6	8	5	10	13	16	14	15	7	3	9	4	11	12	1
2	6	8	15	10	13	16	14	5	7	3	9	4	11	12	1
2	6	16	15	10	13	12	14	5	7	3	9	4	11	8	1
2	15	16	14	10	13	12	6	5	7	3	9	4	11	8	1
16	15	13	14	10	9	12	6	5	7	3	2	4	11	8	1

Figure 6.15 An example of building a heap bottom up. The numbers on top are the indices. The circled numbers are those that have been exchanged on that step.

Source: [Manber 1989] (6 and 2 in the first row should be swapped).

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem 6 (Theorem 6.1)**.** *Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.*

Proof idea: there must be at least $n!$ leaves, one for each possible outcome.

/* Recall Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$. The height of the decision tree must be at least $\log(n!)$, i.e., $\Omega(n \log n)$. */

Is the lower bound contradictory to the time complexity of radix sort?

4 Order Statistics

Order Statistics: Minimum and Maximum

Problem 7. *Find the maximum and minimum elements in a given sequence.*

- The obvious solution requires $(n-1) + (n-2)$ $(= 2n-3)$ comparisons between elements.
- Can we do better? Which comparisons could have been avoided?

Order Statistics: Kth-Smallest

Problem 8. *Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.*

Order Statistics: Kth-Smallest (cont.)

procedure Select $(Left, Right, k)$;
begin
  if $Left = Right$ **then**
    $Select := Left$
  else $Partition(X, Left, Right)$;
    let Middle be the output of Partition;
    if $Middle - Left + 1 \geq k$ **then**
      $Select(Left, Middle, k)$
    else
      $Select(Middle + 1, Right, k - (Middle - Left + 1))$
end

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

procedure Select $(Left, Right, k)$;
begin
  if $Left = Right$ **then**
    $Select := Left$
  else $Partition(X, Left, Right)$;
    let Middle be the output of Partition;
    if $Middle \geq k$ **then**
      $Select(Left, Middle, k)$
    else
      $Select(Middle + 1, Right, k)$
end

Order Statistics: Kth-Smallest (cont.)

Algorithm Selection (X, n, k);
begin
  if $(k < 1)$ or $(k > n)$ **then** *print "error"*
  else $S := Select(1, n, k)$
end

5 Finding a Majority

Finding a Majority

Problem 9. *Given a sequence of numbers, find the majority in the sequence or determine that none exists.*

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

/* If there is a majority, it is also a majority of the other $n - 2$ numbers. */

What if they are equal?

Finding a Majority (cont.)

```
Algorithm Majority (X, n);
begin
    C := X[1];  M := 1;
    for i := 2 to n do
        if M = 0 then
            C := X[i];  M := 1
        else
            if C = X[i] then M := M + 1
            else M := M − 1;
```

Finding a Majority (cont.)

```
    if M = 0 then Majority := −1
    else
        Count := 0;
        for i := 1 to n do
            if X[i] = C then Count := Count + 1;
        if Count > n/2 then Majority := C
        else Majority := −1
end
```