Homework 2

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Question1

REFINE THE DEFINITION!!!!!!
我知道題目的要求藏在行文當中沒有很明顯，但沒有寫的還是 -10 分

1. The empty tree, denoted $\perp$, is a binary search tree.
2. If t_l and t_r are BST,
   and every key value (of descendants) in t_l is smaller than k,
   and every key value (of descendants) in t_r is larger than k,
   then $node(k, t_l, t_r)$ is also a BST.

Question1

又或者是利用 *Min* 與 *Max*

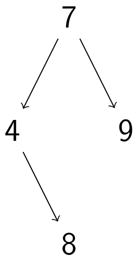
2. If t_l and t_r are BST,
   and $Max(t_l) < k < Min(t_r)$
   then $node(k, t_l, t_r)$ is also a BST.
   其中

$$Min(t) = \begin{cases} \infty, & t = \bot \\ min(k, Min(t_l), Min(t_r)), & otherwise \end{cases}$$

$$Max(t) = \begin{cases} 0, & t = \bot \\ max(k, Max(t_l), Max(t_r)), & otherwise \end{cases}$$

Question1

一樣不扣少寫 search 的分
一樣沒寫 base case 扣 2 分
常錯的點：沿用作業一的定義，只看子樹的樹根，-5 分
以下面這個樹為例子，$4 < 8$, $4 < 7$, $7 < 9$，但是
inorder traverse: $4 \rightarrow 8 \rightarrow 7 \rightarrow 9$, 沒有照順序！
應該要看「整棵」左樹/右樹



Question1

要注意的點：忽略 empty tree 的情境
但是若是寫「every/for each」node value in t_l/t_r，算對
邏輯上 for all，若沒有的話就是 true

要注意的點：$\perp$ 與 $node(k, t_l, t_r)$ 雖然都代表 tree，但卻是兩種不同的東西，並不存在「$node(k, t_l, t_r) = \perp$」這種東西。不扣分

若用了正確的邏輯但不是用 inductive 的方式定義，扣 4 分

Question1

$$Rank(t, n) = \begin{cases} 0, t = \bot or\ not\ Exist(t, n) \\ Rank'(t, n), otherwise \end{cases}$$

$$Exist(t, n) = \begin{cases} false, t = \bot \\ true, t = node(n, t_l, t_r) \\ Exist(t_l, n), t = node(k, t_l, t_r)\ and\ n < k \\ Exist(t_r, n), t = node(k, t_l, t_r)\ and\ n > k \end{cases}$$

$$Rank'(node(k, t_l, t_r), n) = \begin{cases} Rank'(t_l, n), n < k \\ Count(t_l) + 1, n = k \\ Count(t_l) + 1 + Rank'(t_r, n), n > k \end{cases}$$

$$Count(t) = \begin{cases} 0, t = \bot \\ Count(t_l) + 1 + Count(t_r), t = node(k, t_l, t_r) \end{cases}$$

Question1

或者也能這麼寫

$$Rank(t, n) = Rank'(t, n, 0)$$

$$Rank'(t, n, x) = \begin{cases} 0, t = \bot \\ Rank'(t_l, n, x), t = node(k, t_l, t_r) \text{ and } n < k \\ x + Count(t_l) + 1, t = node(k, t_l, t_r) \text{ and } n = k \\ Rank'(t_r, n, x + Count(t_l) + 1), otherwise \end{cases}$$

$$Count(t) = \begin{cases} 0, t = \bot \\ Count(t_l) + 1 + Count(t_r), t = node(k, t_l, t_r) \end{cases}$$

Question1

注意不要寫 code 或 pseudocode，因為第一次助教課有提過，作業二這裡先扣 5 分（10 分的一半）

但書：代碼只有一連串 if/else if 與純粹的 return，沒有 assignment、指標等成分的，不會扣分，不過可以試著改寫成數學函數

定義函數要寫清楚，包括用到的任何其他函數（算出樹的 size 的函數、偵測一個值是否在樹裡頭的函數），也要一併定義，不然扣 3 分

Question2

Differing by k bits is impossible, proof of this is trivial.
Differing by $k-1$ bits :

1. Construct a Gray code for n objects.
2. Complement the string every two code.
3. Add leading 0 or 1 alternatively to avoid duplicated strings.

Differing by $k-i$ bits :

Revise step 1 : Construct a Gray code which differs by i bits for n objects.

cont'd

Example : $(n = 6)$

1. $k = 3$

   000 001 011
   100 110 111

---

2. $k = 3$

   000 110 011
   011 110 000

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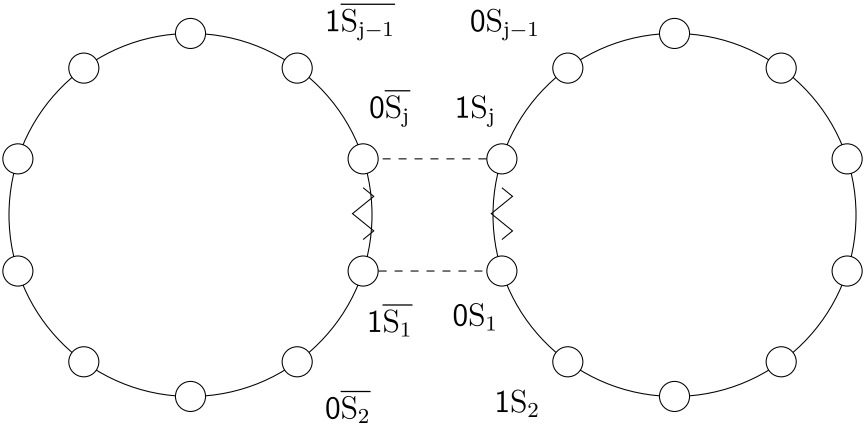
3. $k = 4$

   0000 1110 0011
   1011 0110 1000

cont'd (similar to the lecture)



$S_1, S_2, \ldots, S_j$ is an anti-Gray code(Differ by $k-1$).

cont'd (observation and innovation)

Induction from $\frac{n}{2}$ to n :
如上頁圖，已知 $S_1, S_2, ..., S_j$ 是 $\frac{n}{2}$ 個 objects 的 anti-Gray code

1. Complement 左邊
2. 在兩邊第一個 bit 交錯插入 0 和 1

得到 n 個 objects 的 anti-Gray code (Differ by at least $k-1$)

cont’d

注意

1. 先 construct $\frac{n}{2}$ 個 objects 的 Gray code 會造成如果 $\frac{n}{2}$ 是奇數，則 Gray code is open. (Differ by more than 1 bit)，再 Complement 會相差小於 $k-1$ 個 bits。Consider the case $n=6$ (6 objects).
2. 注意 complement 時有沒有重複的 string。

Question3

[Base Case] (h=2) $T(2) = T(1) + T(0) + 1 = 0 + 1 + 1 = 2$

[Induction Hypothesis] $T(h) = F(h+2) - 1$

$$T(2) = F(4) - 1 = 3 - 1 = 2$$

[Induction Step]

$$\begin{aligned} T(h+1) &= T(h) + T(h-1) + 1 \\ &= (F(h+2) - 1) + (F(h+1) - 1) + 1 \\ &= F(h+2) + F(h+1) - 1 \\ &= F(h+3) - 1 \end{aligned}$$

Question4

[Base Case] height=0

[Induction Hypothesis] height=h+1

$$(2 * \textit{Sum of the height in } T) + \textit{height of root}$$
$$= 2 * (2^{h+1} - h - 2) \; + \; (h + 1)$$
$$= 2^{h+2} - 2h - 4 + h + 1$$
$$= 2^{(h+1)+1} - (h + 1) - 2$$

Question5

[Base Case] $\begin{cases} x = n \text{ (assign n to x)} \\ y = 0 \text{ (assign zero to y)} \end{cases}$

[Induction Hypothesis] $n^2 = x^2 + y \ (x > 0)$

x	3	5	8	9
y	0	2	1	0

[Induction Step]

$$\begin{cases} n' = n \\ x' = x - 1 \Longrightarrow x = x' + 1 \\ y' = y + 2x - 1 \Longrightarrow y = y' - 2x + 1 \\ \qquad\qquad\qquad\qquad = y' - 2(x' + 1) + 1 = y' - 2x' - 1 \end{cases}$$

Question5(Continue)

[Induction Step]

$$\begin{cases} n' = n \\ x' = x - 1 \Longrightarrow x = x' + 1 \\ y' = y + 2x - 1 \Longrightarrow y = y' - 2x + 1 \\ \qquad\qquad\qquad\qquad = y' - 2(x' + 1) + 1 = y' - 2x' - 1 \end{cases}$$

$$\begin{aligned} & n' = x^2 + y \text{ (by Induction Hypothesis)} \\ & = (x' + 1)^2 + (y' - 2x' - 1) \\ & = (x'^2 + 2x' + 1) + (y' - 2x' - 1) \\ & = x'^2 + y' \end{aligned}$$