

Data Structures

A Supplement

(Based on [Manber 1989])

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- 🌐 A (max) heap is a **binary tree** whose keys satisfy the heap property:
the key of every node is greater than or equal to the key of any of its children.
- 🌐 It supports the two basic operations of a **priority queue**:

Heaps

- 🌐 A (max) heap is a **binary tree** whose keys satisfy the heap property:
the key of every node is greater than or equal to the key of any of its children.
- 🌐 It supports the two basic operations of a **priority queue**:
 - ☀️ *Insert*(x): insert the key x into the heap.
 - ☀️ *Remove*() : remove and return the largest key from the heap.

Heaps (cont.)

-  A binary tree can be represented implicitly by an array A as follows:
1. The root is stored in $A[1]$.
 2. The **left child** of $A[i]$ is stored in $A[2i]$ and the **right child** is stored in $A[2i + 1]$.

Heaps (cont.)

Algorithm Remove_Max_from_Heap (A, n);
begin

if $n = 0$ **then** print “the heap is empty”
 else $Top_of_the_Heap := A[1]$;
 $A[1] := A[n]$; $n := n - 1$;
 $parent := 1$; $child := 2$;
 while $child \leq n - 1$ **do**
 if $A[child] < A[child + 1]$ **then**
 $child := child + 1$;
 if $A[child] > A[parent]$ **then**
 $swap(A[parent], A[child])$;
 $parent := child$;
 $child := 2 * child$
 else $child := n$

end

Heaps (cont.)

Algorithm Insert_to_Heap (A, n, x);
begin

$n := n + 1$;

$A[n] := x$;

$child := n$;

$parent := n \text{ div } 2$;

while $parent \geq 1$ **do**

if $A[parent] < A[child]$ **then**

$swap(A[parent], A[child])$;

$child := parent$;

$parent := parent \text{ div } 2$

else $parent := 0$

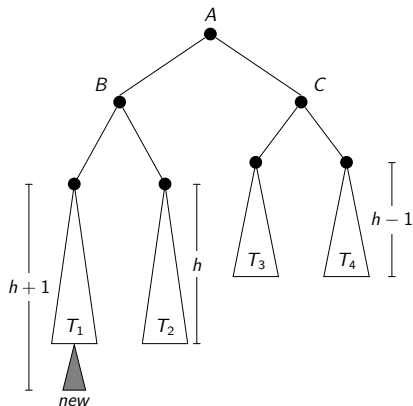
end

Definition

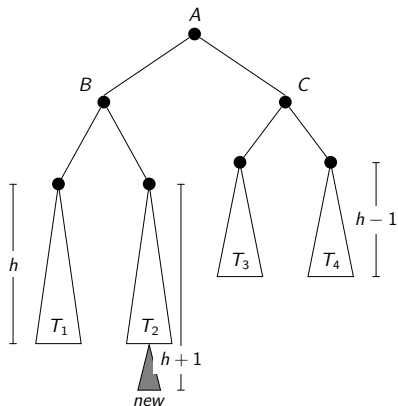
An AVL tree is a binary search tree such that, for every node, the **difference between the heights** of its left and right subtrees is **at most 1** (the height of an empty tree is defined as 0).

This definition guarantees a maximal height of $O(\log n)$ for any AVL tree of n nodes.

AVL Trees (cont.)



(a)

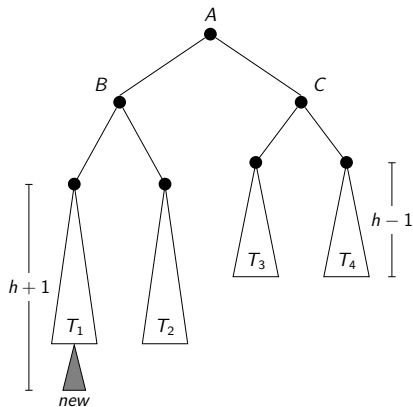


(b)

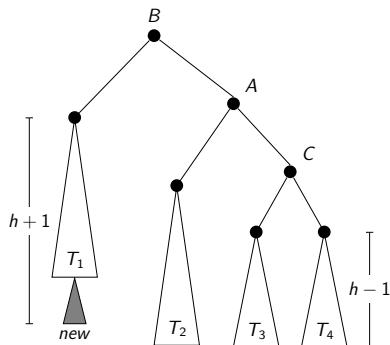
Figure: Insertions that invalidate the AVL property.

Source: redrawn from [Manber 1989, Figure 4.13].

AVL Trees (cont.)



(a)



(b)

Figure: A single rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.14].

AVL Trees (cont.)

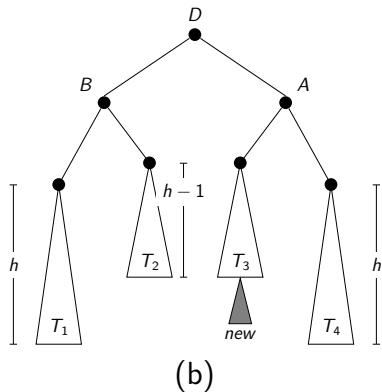
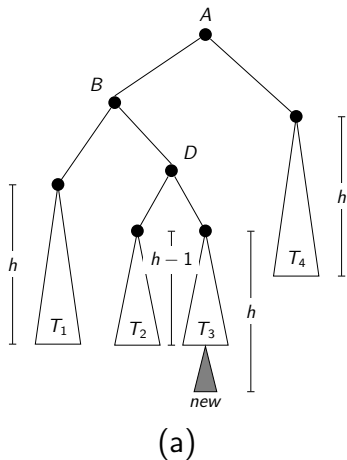


Figure: A double rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.15].

Union-Find

- 🌐 There are n elements x_1, x_2, \dots, x_n divided into groups. Initially, each element is in a group by itself.
- 🌐 Two operations on the elements and groups:
 - ☀️ $find(A)$: returns the name of A 's group.
 - ☀️ $union(A, B)$: combines A 's and B 's groups to form a new group with a unique name.
- 🌐 To tell if two elements are in the same group, one may issue a find operation for each element and see if the returned names are the same.

Union-Find (cont.)

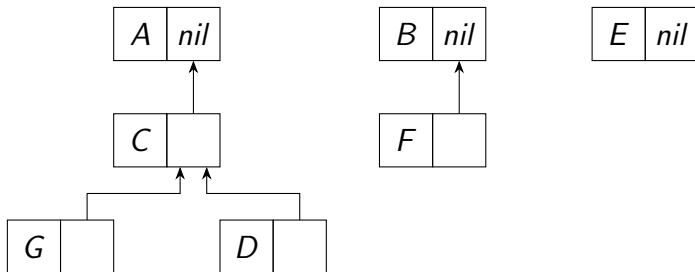


Figure: The representation for the union-find problem.

Source: redrawn from [Manber 1989, Figure 4.16].

Balancing

- 🌍 The root also stores the number of elements in (i.e., the size of) its group.
- 🌍 To *balance* the tree resulted from a union operation, *let the smaller group join the larger group* and update the size of the larger group accordingly.

Theorem (Theorem 4.2)

If balancing is used, then any tree of height h must contain at least 2^h elements.

- 🌍 Any sequence of m find or union operations (where $m \geq n$) takes $O(m \log n)$ steps.

Union-Find (cont.)

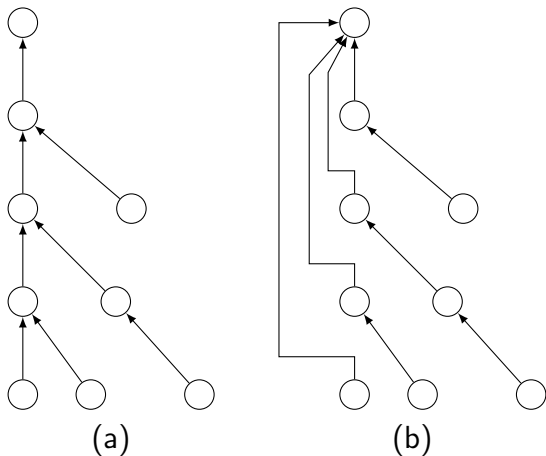


Figure: Path compression: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.17].

Effect of Path Compression

Theorem (Theorem 4.3)

If both balancing and path compression are used, any sequence of m find or union operations (where $m \geq n$) takes $O(m \log^ n)$ steps.*

The value of $\log^* n$ intuitively equals the number of times that one has to apply \log to n to bring its value down to 1.

Code for Union-Find

```
Algorithm Union_Find_Init(A,n);  
begin  
  for i := 1 to n do  
    A[i].parent := nil;  
    A[i].size := 1  
  end  
end
```

```
Algorithm Find(a);  
begin  
  if A[a].parent <> nil then  
    A[a].parent := Find(A[a].parent);  
    Find := A[a].parent;  
  else  
    Find := a  
  end  
end
```


Code for Union-Find (cont.)

```
Algorithm Union(a,b);
begin
  x := Find(a);
  y := Find(b);
  if x <> y then
    if A[x].size > A[y].size then
      A[y].parent := x;
      A[x].size := A[x].size + A[y].size;
    else
      A[x].parent := y;
      A[y].size := A[y].size + A[x].size
    end
  end
end
```