Advanced Graph Algorithms

(Based on [Manber 1989])

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Strongly Connected Components

- A directed graph is *strongly connected* if there is a directed path from every vertex to every other vertex.

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- A directed graph is *strongly connected* if there is a directed path from every vertex to every other vertex.
- A *strongly connected component* (SCC) is a maximal subset of the vertices such that its induced subgraph is strongly connected (namely, there is no other subset that contains it and induces a strongly connected graph).

Strongly Connected Components (cont.)

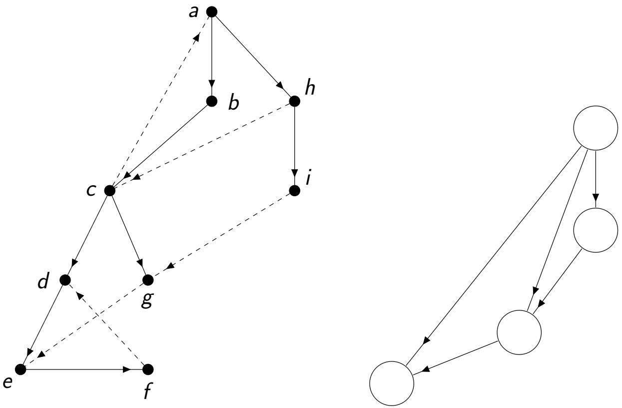

Figure: A directed graph and its strongly connected component graph.

Source: redrawn from [Manber 1989, Figure 7.30].

Strongly Connected Components (cont.)

Lemma (7.11)

Two distinct vertices belong to the same SCC if and only if there is a circuit containing both of them.

Strongly Connected Components (cont.)

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Lemma (7.11)

Two distinct vertices belong to the same SCC if and only if there is a circuit containing both of them.

Lemma (7.12)

Each vertex belongs to exactly one SCC.

Strongly Connected Components (cont.)

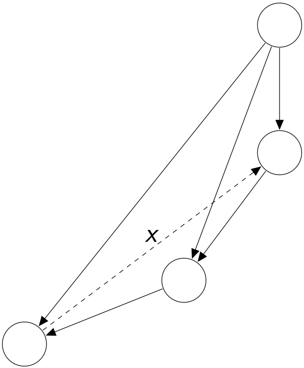

Figure: Adding an edge connecting two different strongly connected components.

Source: redrawn from [Manber 1989, Figure 7.31].

Strongly Connected Components (cont.)

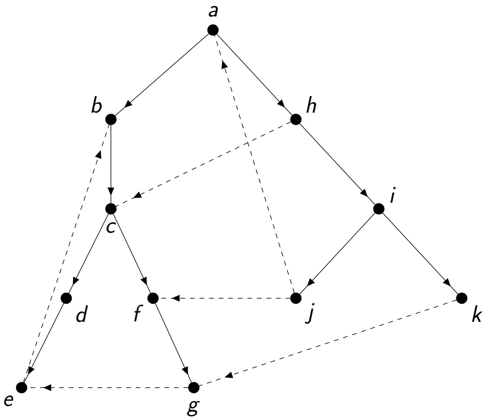

Figure: The effect of cross edges.

Source: redrawn from [Manber 1989, Figure 7.32].

Strongly Connected Components (cont.)

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Algorithm Strongly_Connected_Components(G, n);
begin
  for every vertex v of G **do**
    $v.DFS\_Number := 0$;
    $v.Component := 0$;
  $Current\_Component := 0$; $DFS\_N := n$;
  while $v.DFS\_Number = 0$ for some v **do**
    $SCC(v)$
end

procedure SCC(v);
begin
  $v.DFS\_Number := DFS\_N$;
  $DFS\_N := DFS\_N - 1$;
  insert v into $Stack$;
  $v.High := v.DFS\_Number$;

Strongly Connected Components (cont.)

for all edges (v, w) **do**
  if $w.DFS\_Number = 0$ **then**
    $SCC(w)$;
    $v.High := \max(v.High, w.High)$
  else if $w.DFS\_Number > v.DFS\_Number$
      and $w.Component = 0$ **then**
    $v.High := \max(v.High, w.DFS\_Number)$
    // $\max(v.High, w.High)$ also works
if $v.High = v.DFS\_Number$ **then**
  $Current\_Component := Current\_Component + 1$;
  repeat
    remove x from the top of $Stack$;
    $x.component := Current\_Component$
  until $x = v$
end

Strongly Connected Components (cont.)

for all edges (v, w) **do**
  if $w.DFS\_Number = 0$ **then**
    $SCC(w)$;
    $v.High := \max(v.High, w.High)$
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  $Current\_Component := Current\_Component + 1$;
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    remove x from the top of $Stack$;
    $x.component := Current\_Component$
  until $x = v$
end

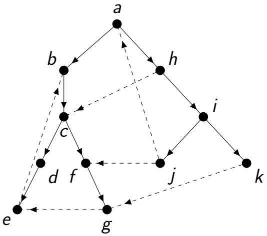
Time complexity:

Strongly Connected Components (cont.)

```
for all edges (v, w) do
    if w.DFS_Number = 0 then
        SCC(w);
        v.High := max(v.High, w.High)
    else if w.DFS_Number > v.DFS_Number
              and w.Component = 0 then
            v.High := max(v.High, w.DFS_Number)
            // max(v.High, w.High) also works
if v.High = v.DFS_Number then
    Current_Component := Current_Component + 1;
    repeat
        remove x from the top of Stack;
        x.component := Current_Component
    until x = v
end
```

Time complexity: $O(|E| + |V|)$.

Strongly Connected Components (cont.)



	a	b	c	d	e	f	g	h	i	j	k
	11	10	9	8	7	6	5	4	3	2	1
a	11	-	-	-	-	-	-	-	-	-	-
b	11	10	-	-	-	-	-	-	-	-	-
c	11	10	9	-	-	-	-	-	-	-	-
d	11	10	9	8	-	-	-	-	-	-	-
e	11	10	9	8	10	-	-	-	-	-	-
d	11	10	9	10	10	-	-	-	-	-	-
c	11	10	10	10	10	-	-	-	-	-	-
f	11	10	10	10	10	6	-	-	-	-	-
g	11	10	10	10	10	6	7	-	-	-	-
f	11	10	10	10	10	7	7	-	-	-	-
c	11	10	10	10	10	7	7	-	-	-	-
b	11	10	10	10	10	7	7	-	-	-	-
a	11	10	10	10	10	7	7	-	-	-	-
h	11	10	10	10	10	7	7	4	-	-	-
i	11	10	10	10	10	7	7	4	3	-	-
j	11	10	10	10	10	7	7	4	3	11	-
i	11	10	10	10	10	7	7	4	11	11	-
k	11	10	10	10	10	7	7	4	11	11	1
i	11	10	10	10	10	7	7	4	11	11	1
h	11	10	10	10	10	7	7	11	11	11	1
a	11	10	10	10	10	7	7	11	11	11	1

Figure: An example of computing *High* values and strongly connected components.

Source: redrawn from [Manber 1989, Figure 7.34].

Odd-Length Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle of odd length.

Odd-Length Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle of odd length.

- A cycle must reside completely within a strongly connected component (SCC), so we exam each SCC separately.
- Mark the nodes of an SCC with "even" or "odd" using DFS.
- If we have to mark a node that is already marked in the opposite, then we have found an odd-length cycle.

Biconnected Components

- An undirected graph is *biconnected* if there are at least two vertex-disjoint paths from every vertex to every other vertex.

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- A graph is *not* biconnected if and only if there is a vertex whose removal disconnects the graph. Such a vertex is called an *articulation point*.

Biconnected Components

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- A graph is *not* biconnected if and only if there is a vertex whose removal disconnects the graph. Such a vertex is called an *articulation point*.
- A *biconnected component* (BCC) is a *maximal* subset of the edges such that its induced subgraph is biconnected (namely, there is no other subset that contains it and induces a biconnected graph).

Biconnected Components (cont.)

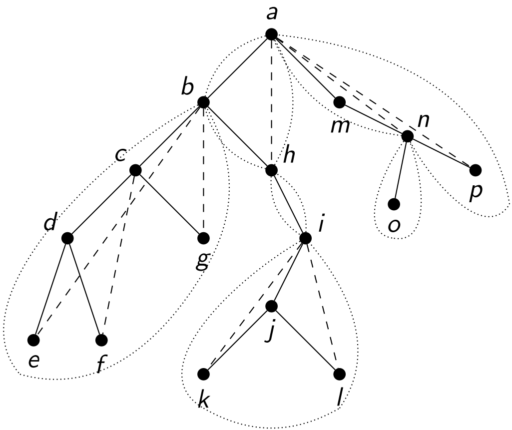

Figure: The structure of a nonbiconnected graph.

Source: redrawn from [Manber 1989, Figure 7.25].

Biconnected Components (cont.)

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Lemma (7.9)

Two distinct edges e and f belong to the same BCC if and only if there is a cycle containing both of them.

Biconnected Components (cont.)

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Lemma (7.9)

Two distinct edges e and f belong to the same BCC if and only if there is a cycle containing both of them.

Lemma (7.10)

Each edge belongs to exactly one BCC.

Biconnected Components (cont.)

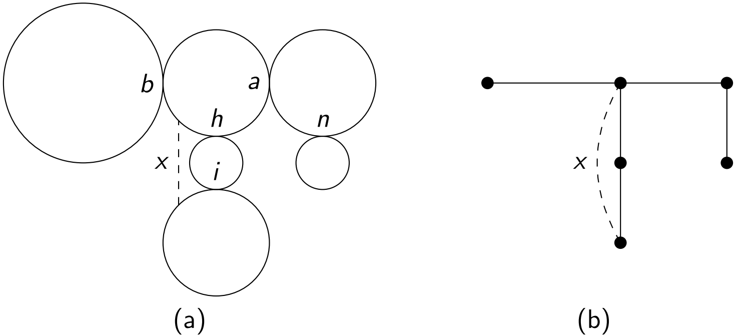


Figure: An edge that connects two different biconnected components. (a) The components corresponding to the graph of Figure 7.25 with the articulation points indicated. (b) The biconnected component tree.

Source: redrawn from [Manber 1989, Figure 7.26].

Biconnected Components (cont.)

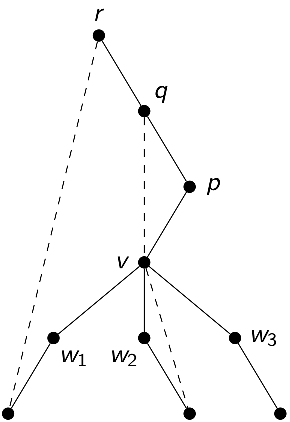


Figure: Computing the *High* values.

Source: redrawn from [Manber 1989, Figure 7.27].

Biconnected Components (cont.)

Algorithm Biconnected_Components(G, v, n);
begin
  for every vertex w **do** $w.DFS\_Number := 0$;
  $DFS\_N := n$;
  $BC(v)$
end

procedure BC(v);
begin
  $v.DFS\_Number := DFS\_N$;
  $DFS\_N := DFS\_N - 1$;
  insert v into $Stack$;
  $v.High := v.DFS\_Number$;

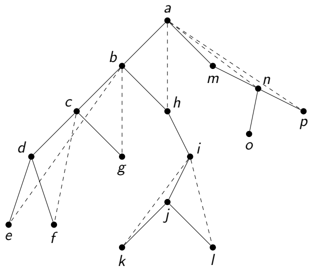
Biconnected Components (cont.)

```
for all edges (v, w) do
    insert (v, w) into Stack;
    if w is not the parent of v then
        if w.DFS_Number = 0 then
            BC(w);
            if w.High ≤ v.DFS_Number then
                remove all edges and vertices
                    from Stack until v is reached;
                insert v back into Stack;
            v.High := max(v.High, w.High)
        else
            v.High := max(v.High, w.DFS_Number)
            // max(v.High, w.High) would not work, unlike in SCC
end
```

Biconnected Components (cont.)

procedure **BC**(v);
begin
  $v.DFS\_Number := DFS\_N$;
  $DFS\_N := DFS\_N - 1$;
  $v.High := v.DFS\_Number$;
  for all edges (v, w) **do**
    if w is not the parent of v **then**
      insert (v, w) into $Stack$;
      if $w.DFS\_Number = 0$ **then**
        $BC(w)$;
        if $w.high \leq v.DFS\_Number$ **then**
          remove all edges from $Stack$
            until (v, w) is reached;
        $v.High := \max(v.High, w.High)$
      else
        $v.High := \max(v.High, w.DFS\_Number)$
end

Biconnected Components (cont.)



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
a	16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
b	16	15	-	-	-	-	-	-	-	-	-	-	-	-	-	-
c	16	15	14	-	-	-	-	-	-	-	-	-	-	-	-	-
d	16	15	14	13	-	-	-	-	-	-	-	-	-	-	-	-
e	16	15	14	13	15	-	-	-	-	-	-	-	-	-	-	-
d	16	15	14	15	15	-	-	-	-	-	-	-	-	-	-	-
f	16	15	14	15	15	14	-	-	-	-	-	-	-	-	-	-
d	16	15	14	15	15	14	-	-	-	-	-	-	-	-	-	-
c	16	15	15	15	15	14	-	-	-	-	-	-	-	-	-	-
g	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
c	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
b	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
h	16	15	15	15	15	14	15	16	-	-	-	-	-	-	-	-
i	16	15	15	15	15	14	15	16	8	-	-	-	-	-	-	-
j	16	15	15	15	15	14	15	16	8	7	-	-	-	-	-	-
k	16	15	15	15	15	14	15	16	8	7	8	-	-	-	-	-
j	16	15	15	15	15	14	15	16	8	8	8	-	-	-	-	-
l	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
j	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
i	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
h	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
b	16	16	15	15	15	14	15	16	8	8	8	8	-	-	-	-
a	16	16	15	15	15	14	15	16	8	8	8	8	-	-	-	-
m	16	16	15	15	15	14	15	16	8	8	8	8	4	-	-	-
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	-	-
o	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	-
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	-
p	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
m	16	16	15	15	15	14	15	16	8	8	8	8	16	16	2	16
a	16	16	15	15	15	14	15	16	8	8	8	8	16	16	2	16

Figure: An example of computing the *High* values and biconnected components.

Source: redrawn from [Manber 1989, Figure 7.29].

Even-Length Cycles

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Problem

Given a connected undirected graph $G = (V, E)$, determine whether it contains a cycle of even length.

Even-Length Cycles

Problem

Given a connected undirected graph $G = (V, E)$, determine whether it contains a cycle of even length.

Theorem

Every biconnected graph that has more than one edge and is not merely an odd-length cycle contains an even-length cycle.

Even-Length Cycles (cont.)

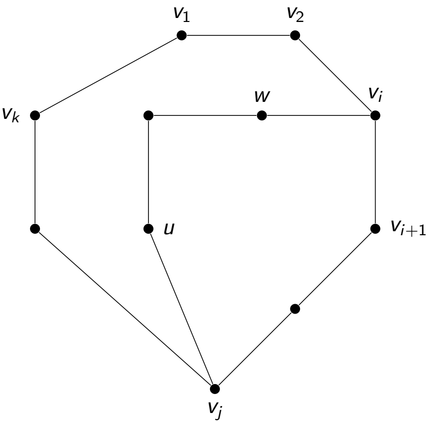

Figure: Finding an even-length cycle.

Source: redrawn from [Manber 1989, Figure 7.35].

Network Flows

- Consider a directed graph, or network, $G = (V, E)$ with two distinguished vertices: s (the source) with indegree 0 and t (the sink) with outdegree 0.
- Each edge e in E has an associated positive weight $c(e)$, called the *capacity* of e.

Network Flows (cont.)

- A **flow** is a function f on E that satisfies the following two conditions:
  1. $0 \leq f(e) \leq c(e)$.
  2. $\displaystyle\sum_{u} f(u,v) = \sum_{w} f(v,w)$, for all $v \in V - \{s,t\}$.
- The **network flow problem** is to maximize the flow f for a given network G.

Network Flows (cont.)

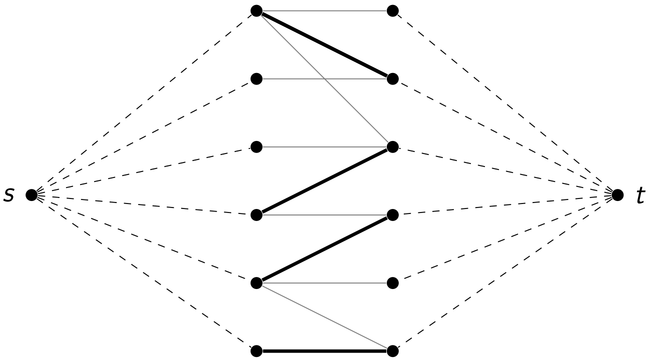

Figure: Reducing bipartite matching to network flow (the directions of all the edges are from left to right).

Source: redrawn from [Manber 1989, Figure 7.39].

Augmenting Paths

- An **augmenting path** w.r.t. a given flow f (of a network G) is a directed path from s to t consisting of edges from G, but not necessarily in the same diretion; each of these edges (v, u) satisfies exactly one of:
  1. (v, u) is in the same direction as it is in G, and $f(v, u) < c(v, u)$. (*forward edge*)
  2. (v, u) is in the opposite direction in G (namely, $(u, v) \in E$), and $f(u, v) > 0$. (*backward edge*)
- If there exists an augmenting path w.r.t. a flow f (f *admits* an augmenting path), then f is not maximum.

Augmenting Paths (cont.)

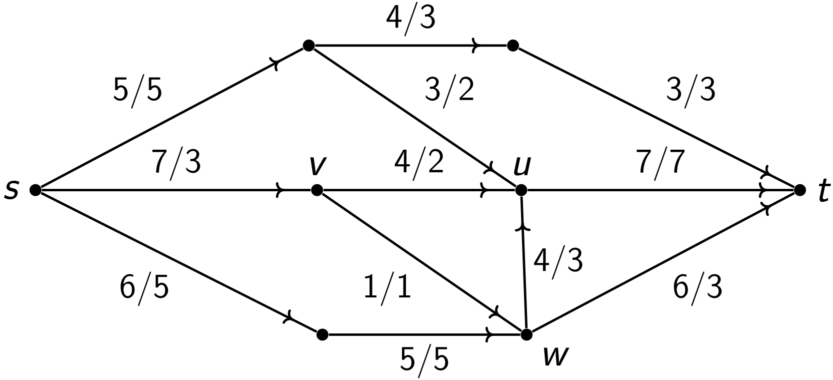

Figure: An example of a network with a (nonmaximum) flow.

Source: redrawn from [Manber 1989, Figure 7.40].

Augmenting Paths (cont.)

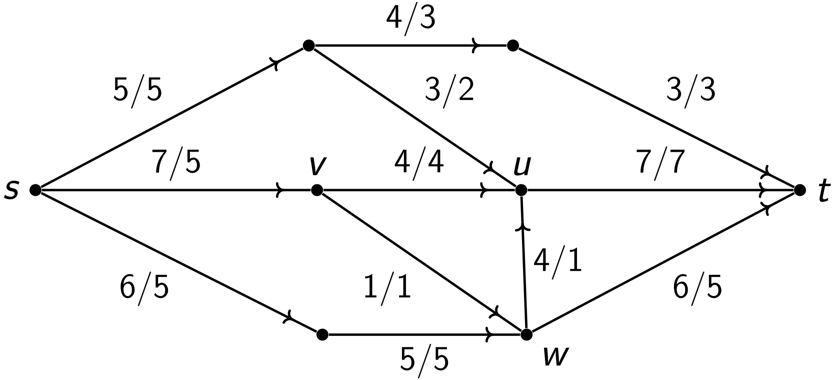

Figure: The result of augmenting the flow of Figure 7.40.

Source: redrawn from [Manber 1989, Figure 7.41].

Properties of Network Flows

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Theorem (Augmenting-Path)

A flow f is maximum if and only if it admits no augmenting path.

A *cut* is a set of edges that separate s from t, or more precisely a set of the form $\{(v, w) \in E \mid v \in A \text{ and } w \in B\}$, where $B = V - A$ such that $s \in A$ and $t \in B$.

Theorem (Max-Flow Min-Cut)

The value of a maximum flow in a network is equal to the minimum capacity of a cut.

Properties of Network Flows (cont.)

Theorem (Integral-Flow)

If the capacities of all edges in the network are integers, then there is a maximum flow whose value is an integer.

Residual Graphs

- The **residual graph** with respect to a network $G = (V, E)$ and a flow f is the network $R = (V, F)$, where F consists of all forward and backward edges and their capacities are given as follows:
  1. $c_R(v, w) = c(v, w) - f(v, w)$ if (v, w) is a forward edge and
  2. $c_R(v, w) = f(w, v)$ if (v, w) is a backward edge.
- An augmenting path is thus a regular directed path from s to t in the residual graph.

Residual Graphs (cont.)

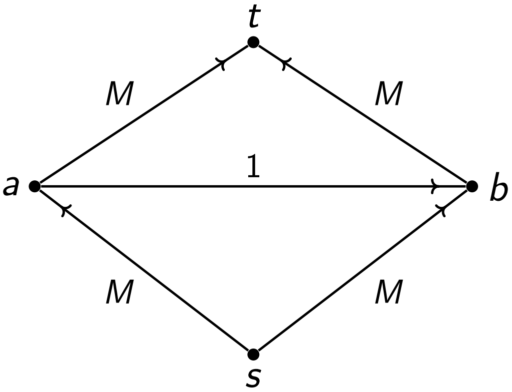

Figure: A bad example of network flow.

Source: redrawn from [Manber 1989, Figure 7.42].