Homework Assignment #1

Note

This assignment is due 2:10PM Tuesday, September 24, 2019. Please write or type your answers on A4 (or similar size) paper. Drop your homework by the due time in Yih-Kuen Tsay's mail box on the first floor of Management College Building 2. Late submission will be penalized by 20% for each working day overdue. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

There are five problems in this assignment, each accounting for 20 points. You must use *induction* for all proofs. (Note: problems marked with "(X.XX)" are taken from [Manber 1989] with probable adaptation.)

1. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$, for all $n \geq 1$.

2. The Harmonic series $H(k)$ is defined by $H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}$. Prove that $H(2^n) \geq 1 + \frac{n}{2}$, for all $n \geq 0$ (which implies that $H(k)$ diverges).

3. (2.7) Given a set of $n+1$ numbers out of the first $2n$ (starting from 1) natural numbers 1, 2, 3, …, $2n$, prove that there are two numbers in the set, one of which divides the other.

4. (2.37) Consider the recurrence relation for Fibonacci numbers $F(n) = F(n-1) + F(n-2)$. Without solving this recurrence, compare $F(n)$ to $G(n)$ defined by the recurrence $G(n) = G(n-1) + G(n-2) + 1$. It seems obvious that $G(n) > F(n)$ (because of the extra 1). Yet the following is a seemingly valid proof (by induction) that $G(n) = F(n) - 1$. We assume, by induction, that $G(k) = F(k) - 1$ for all k such that $1 \leq k \leq n$, and we consider $G(n+1)$:

$$G(n+1) = G(n) + G(n-1) + 1 = F(n) - 1 + F(n-1) - 1 + 1 = F(n+1) - 1$$

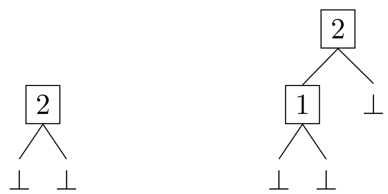
   What is wrong with this proof?

5. The set of all binary trees that store non-negative integer key values may be defined inductively as follows.

   (a) The empty tree, denoted $\perp$, is a binary tree.

   (b) If t_l and t_r are binary trees, then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$, is also a binary tree.

   So, for instance, $node(2, \perp, \perp)$ is a single-node binary tree storing key value 2 and $node(2, node(1, \perp, \perp), \perp)$ is a binary tree with two nodes — the root and its left child, storing key values 2 and 1 repsectively. Pictorially, they may be depicted as below.



(a) (5 points) Define inductively a function SUM that computes the sum of all key values of a binary tree. Let $SUM(\bot) = 0$, though the empty tree does not store any key value.

(b) (5 points) Suppose, to differentiate the empty tree from a non-empty tree whose key values sum up to 0, we require that $SUM(\bot) = -1$. Give another definition for SUM that meets this requirement; again, induction should be used somewhere in the definition.

(c) (10 points) Refine the definition of binary trees so that it defines the set of all binary *search* trees where the key value stored in the left child of an internal node is smaller than that of the internal node and the key value stored in the right child is larger.