Algorithms 2020: Basic Graph Algorithms

(Based on [Manber 1989])

Yih-Kuen Tsay

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1 Introduction

The Königsberg Bridges Problem

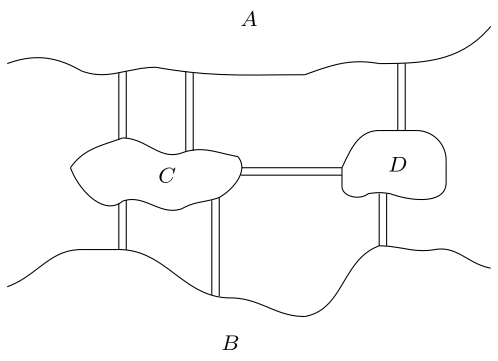


Figure: The Königsberg bridges problem.
Source: redrawn from [Manber 1989, Figure 7.1].

Can one start from one of the lands, cross every bridge exactly once, and return to the origin?

The Königsberg Bridges Problem (cont.)

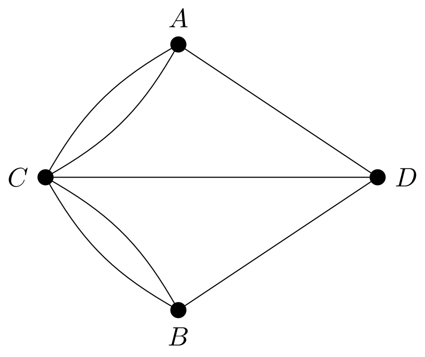


Figure: The graph corresponding to the Königsberg bridges problem.
Source: redrawn from [Manber 1989, Figure 7.2].

Graphs

- A graph consists of a set of vertices (or nodes) and a set of edges (or links, each normally connecting two vertices).
- A graph is commonly denoted as $G(V, E)$, where
  - G is the name of the graph,
  - V is the set of vertices, and
  - E is the set of edges.

Graphs (cont.)

- Undirected vs. Directed Graph
- Simple Graph vs. Multigraph

  /* In a multigraph, multiple edges are allowed between a pair of vertices; the edges are not labeled (and thus cannot be distinguished). */
- Path, Simple Path, Trail

  /* Path often really means simple path (also called open path), where all vertices are distinct. Trail is just another name for path, but strongly suggests that it may be a cycle (also called closed path, where the start and the end vertices are the only repeated vertices). */
- Circuit, Cycle
- Degree, In-Degree, Out-Degree
- Connected Graph, Connected Components
- Tree, Forest
- Subgraph, Induced Subgraph

  /* A vertex-induced subgraph must include every edge in the original graph that connects a pair of the selected vertices. */
- Spanning Tree, Spanning Forest
- Weighted Graph

Modeling with Graphs

- Reachability
  - Finding program errors

    /* A program state corresponds to a vertex and there is a directed edge from one vertex to another if the program represented by the first vertex may (in one execution step) become the program state represented by the second vertex. */
  - Solving sliding tile puzzles

    /* A configuration of the sliding tiles corresponds to a vertex and there is a directed edge from one vertex to another if the configuration represented by the first vertex may (in one sliding step) become the configuration represented by the second vertex. */
- Shortest Paths

  - Finding the fastest route to a place
  - Routing messages in networks

- Graph Coloring
  - Coloring maps
  - Scheduling classes

    /* A class corresponds to a vertex and there is an undirected edge between two vertices if the two classes represented by the two vertices are taught by the same instructor. The colors represent the time slots.

    Another interpretation: There is an undirected edge between two vertices if there is a time conflict between the two classes represented by the two vertices. The colors represent the classrooms. */

Eulerian Graphs

Problem 1. *Given an undirected connected graph $G = (V, E)$ such that all the vertices have even degrees, find a circuit P such that each edge of E appears in P exactly once.*

The circuit P in the problem statement is called an *Eulerian circuit*.

Theorem 2. *An undirected connected graph has an Eulerian circuit if and only if all of its vertices have even degrees.*

/* Proof sketch:

(The "only if" part) Suppose the graph has an Eulerian circuit. Each time the circuit enters a vertex, it must also leave the vertex from a different edge. For the first vertex in the circuit, it is left first and entered at last via a different edge. So, every vertex mustt have an even degree.

(The "if" part) The proof is by induction on the number of edges. Note that the graph must contain at least a simple cycle, as every vertex is of an even degree.

Base case: the graph is a simple cycle (with one edge or more). The cycle clearly is an Eulerian circuit.

Inductive step: Remove a simple cycle from the graph. The remaining part of the graph may consist of several separated components. Each component is connected and every vertex in the component also has an even degree. The induction hypothesis applies to each component. Connecting the removed cycle and the Eulerian circuit of each component, we have an Eulerian circuit for the entire graph. */

2 Depth-First Search

Depth-First Search

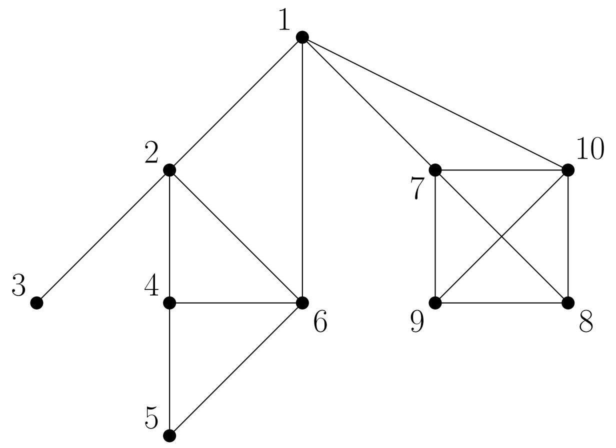


Figure: A DFS for an undirected graph.

Source: redrawn from [Manber 1989, Figure 7.4].

Depth-First Search (cont.)

```
Algorithm Depth_First_Search($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            $Depth\_First\_Search(G, w)$;
        perform postWORK for $(v, w)$
end
```

Depth-First Search (cont.)

```
Algorithm Refined_DFS($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            $Refined\_DFS(G, w)$;
        perform postWORK for $(v, w)$;
    perform postWORK_II on $v$
end
```

Connected Components

```
Algorithm Connected_Components($G$);
begin
    $Component\_Number := 1$;
    while there is an unmarked vertex $v$ do
        $Depth\_First\_Search(G, v)$
        (preWORK:
            $v.Component := Component\_Number$);
        $Component\_Number := Component\_Number + 1$
end
```

Time complexity: $O(|E| + |V|)$.

/* Each edge of the input graph is checked twice (once from each end). The algorithm also has to scan possibly many isolated vertices (in some cases, $|V|$ may be larger than $|E|$). */

DFS Numbers

```
Algorithm DFS_Numbering($G, v$);
begin
    $DFS\_Number := 1$;
    $Depth\_First\_Search(G, v)$
    (preWORK:
        $v.DFS := DFS\_Number$;
        $DFS\_Number := DFS\_Number + 1$)
end
```

Time complexity: $O(|E|)$ (assuming the input graph is connected).

The DFS Tree

```
Algorithm Build_DFS_Tree($G, v$);
begin
    $Depth\_First\_Search(G, v)$
    (postWORK:
        if $w$ was unmarked then
            add the edge $(v, w)$ to $T$);
end
```

The DFS Tree (cont.)

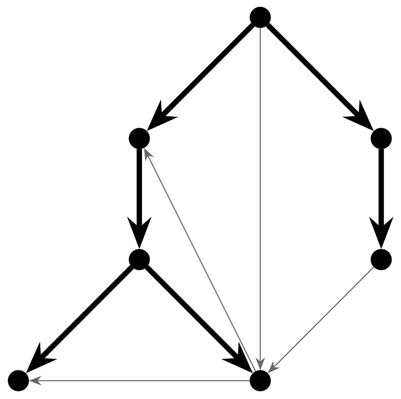


Figure: A DFS tree for a directed graph.
Source: redrawn from [Manber 1989, Figure 7.9].

The DFS Tree (cont.)

Lemma 3 (7.2)**.** *For an undirected graph $G = (V, E)$, every edge $e \in E$ either belongs to the DFS tree T, or connects two vertices of G, one of which is the ancestor of the other in T.*

For undirected graphs, DFS avoids cross edges.

Lemma 4 (7.3)**.** *For a directed graph $G = (V, E)$, if (v, w) is an edge in E such that $v.DFS\_Number < w.DFS\_Number$, then w is a descendant of v in the DFS tree T.*

For directed graphs, cross edges must go "from right to left".

Directed Cycles

Problem 5. *Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle.*

Lemma 6 (7.4)**.** *G contains a directed cycle if and only if G contains a back edge (relative to the DFS tree).*

Directed Cycles (cont.)

```
Algorithm Find_a_Cycle($G$);
begin
    $Depth\_First\_Search(G, v)$ /* arbitrary $v$ */
    (preWORK:
        $v.on\_the\_path$ := $true$;
     postWORK:
        if $w.on\_the\_path$ then
            $Find\_a\_Cycle$ := $true$;
            halt;
        if $w$ is the last vertex on $v$'s list then
            $v.on\_the\_path$ := $false$;)
end
```

Directed Cycles (cont.)

```
Algorithm Refined_Find_a_Cycle($G$);
begin
    $Refined\_DFS(G, v)$ /* arbitrary $v$ */
    (preWORK:
        $v.on\_the\_path$ := $true$;
     postWORK:
        if $w.on\_the\_path$ then
            $Refined\_Find\_a\_Cycle$ := $true$;
            halt;
     postWORK_II:
        $v.on\_the\_path$ := $false$)
end
```

3 Breadth-First Search

4 Breadth-First Search

Breadth-First Search

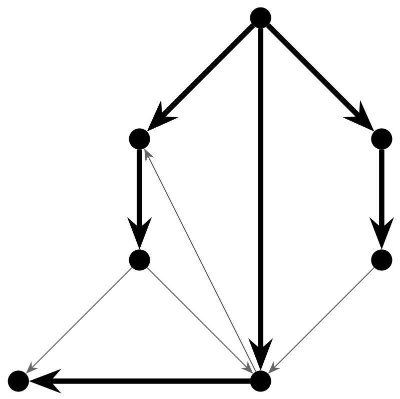


Figure: A BFS tree for a directed graph.

Source: redrawn from [Manber 1989, Figure 7.12].

Breadth-First Search (cont.)

```
Algorithm Breadth_First_Search($G, v$);
begin
    mark $v$;
    put $v$ in a queue;
    while the queue is not empty do
        remove vertex $w$ from the queue;
        perform preWORK on $w$;
        for all edges $(w, x)$ with $x$ unmarked do
            mark $x$;
            add $(w, x)$ to the $BFS$ tree $T$;
            put $x$ in the queue
end
```

Breadth-First Search (cont.)

Lemma 7 (7.5)**.** *If an edge (u, w) belongs to a BFS tree such that u is a parent of w, then u has the minimal BFS number among vertices with edges leading to w.*

Lemma 8 (7.6)**.** *For each vertex w, the path from the root to w in T is a shortest path from the root to w in G.*

Lemma 9 (7.7)**.** *If an edge (v, w) in E does not belong to T and w is on a larger level, then the level numbers of w and v differ by at most* 1*.*

Breadth-First Search (cont.)

```
Algorithm Simple_BFS($G, v$);
begin
   put $v$ in $Queue$;
   while $Queue$ is not empty do
      remove vertex $w$ from $Queue$;
      if $w$ is unmarked then
         mark $w$;
         perform preWORK on $w$;
         for all edges $(w, x)$ with $x$ unmarked do
            put $x$ in $Queue$
end
```

Breadth-First Search (cont.)

```
Algorithm Simple_Nonrecursive_DFS($G, v$);
begin
   push $v$ to $Stack$;
   while $Stack$ is not empty do
      pop vertex $w$ from $Stack$;
      if $w$ is unmarked then
         mark $w$;
         perform preWORK on $w$;
         for all edges $(w, x)$ with $x$ unmarked do
            push $x$ to $Stack$
end
```

5 Topological Sorting

Topological Sorting

Problem 10. *Given a directed acyclic graph $G = (V, E)$ with n vertices, label the vertices from 1 to n such that, if v is labeled k, then all vertices that can be reached from v by a directed path are labeled with labels $> k$.*

Lemma 11 (7.8)**.** *A directed acyclic graph always contains a vertex with indegree 0.*

Topological Sorting (cont.)

```
**Algorithm Topological_Sorting**($G$);
    initialize $v.indegree$ for all vertices; /* by $DFS$ */
    $G\_label := 0$;
    **for** $i := 1$ to $n$ **do**
        **if** $v_i.indegree = 0$ **then** put $v_i$ in $Queue$;
    **repeat**
        remove vertex $v$ from $Queue$;
        $G\_label := G\_label + 1$;
        $v.label := G\_label$;
        **for** all edges $(v, w)$ **do**
            $w.indegree := w.indegree - 1$;
            **if** $w.indegree = 0$ **then** put $w$ in $Queue$
    **until** $Queue$ is empty
```

6 Shortest Paths

Single-Source Shortest Paths

Problem 12. *Given a directed graph $G = (V, E)$ and a vertex v, find shortest paths from v to all other vertices of G.*

Shorted Paths: The Acyclic Case

```
**Algorithm Acyclic_Shortest_Paths**($G, v, n$);
{Initially, $w.SP = \infty$, for every node $w$.}
{A topological sort has been performed on $G$, ...}
**begin**
    let $z$ be the vertex labeled $n$;
    **if** $z \neq v$ **then**
        $Acyclic\_Shortest\_Paths(G - z, v, n - 1)$;
        **for** all $w$ such that $(w, z) \in E$ **do**
            **if** $w.SP + length(w, z) < z.SP$ **then**
                $z.SP := w.SP + length(w, z)$
    **else** $v.SP := 0$
**end**
```

The Acyclic Case (cont.)

```
Algorithm Imp_Acyclic_Shortest_Paths($G, v$);
    for all vertices $w$ do $w.SP := \infty$;
    initialize $v.indegree$ for all vertices;
    for $i := 1$ to $n$ do
        if $v_i.indegree = 0$ then put $v_i$ in $Queue$;
    $v.SP := 0$;
    repeat
        remove vertex $w$ from $Queue$;
        for all edges $(w, z)$ do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$;
            $z.indegree := z.indegree - 1$;
            if $z.indegree = 0$ then put $z$ in $Queue$
    until $Queue$ is empty
```

Shortest Paths: The General Case

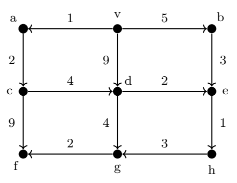
```
Algorithm Single_Source_Shortest_Paths($G, v$);
// Dijkstra's algorithm
begin
    for all vertices $w$ do
        $w.mark := false$;
        $w.SP := \infty$;
    $v.SP := 0$;
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex s.t. $w.SP$ is minimal;
        $w.mark := true$;
        for all edges $(w, z)$ such that $z$ is unmarked do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$
end
```

Time complexity: $O((|E| + |V|) \log |V|)$ (using a min heap).

/* Dijkstra's algorithm assumes that the weight of every edge is non-negative. Given the assumption, going from the source vertex v to some other vertex w will never need to pass any other vertex farther than w. So, the algorithm determines one at a time (the lengths of) the paths to the 1-st, 2-nd, $\cdots$, n-th closest vertices (including the source vertex) from the source vertex.

The main loop requires $O(|V|)$ iterations. In each iteration, there is a delete operation on the heap, which takes $O(\log |V|)$ time. The for loop incurs a total of $O(|E|)$ updates to the heap, each taking $O(\log |V|)$ time. */

The General Case (cont.)



	v	a	b	c	d	e	f	g	h
a	0	1	5	∞	9	∞	∞	∞	∞
c	0	①	5	3	9	∞	∞	∞	∞
b	0	①	5	③	7	∞	12	∞	∞
d	0	①	⑤	③	7	8	12	∞	∞
e	0	①	⑤	③	⑦	8	12	∞	∞
h	0	①	⑤	③	⑦	⑧	12	11	9
g	0	①	⑤	③	⑦	⑧	12	11	⑨
f	0	①	⑤	③	⑦	⑧	12	⑪	⑨

Figure: An example of the single-source shortest-paths algorithm.
Source: redrawn from [Manber 1989, Figure 7.18].

7 Minimum-Weight Spanning Trees

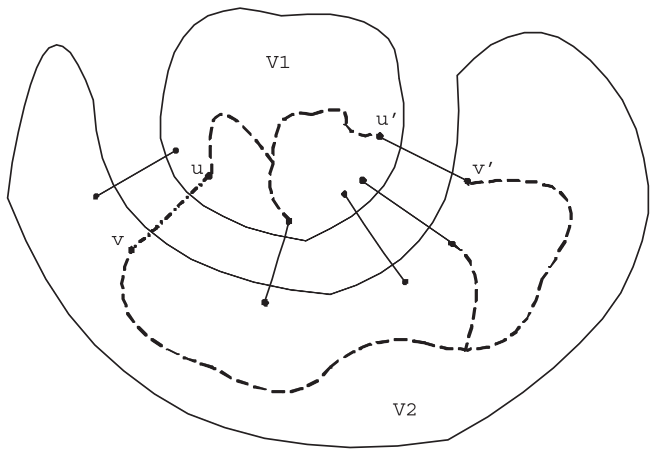
Minimum-Weight Spanning Trees

Problem 13. *Given an undirected connected weighted graph $G = (V, E)$, find a spanning tree T of G of minimum weight.*

Theorem 14. *Let V_1 and V_2 be a partition of V and $E(V_1, V_2)$ be the set of edges connecting nodes in V_1 to nodes in V_2. The edge with the minimum weight in $E(V_1, V_2)$ must be in the minimum-cost spanning tree of G.*

/* Apply the theorem, starting with V_1 containing an arbitrary vertex, and select the minimum-cost edge connecting V_1 to the rest of the graph. Include the connected vertex and we have an expanded V_1 with two vertices. Repeat this process until we cover all vertices of the graph. This is the essence of Prim's algorithm (pseudocode to be given later). */

Minimum-Weight Spanning Trees (cont.)



If $cost(u, v)$ is the smallest among $E(V_1, V_2)$, then $\{u, v\}$ must be in the minimum spanning tree.

/* Suppose $\{u, v\}$ is not chosen. Adding $\{u, v\}$ to the claimed minimum spanning tree will result in a cycle. On the cycle, there must be a heavier $\{u', v'\}$ from $E(V_1, V_2)$. Removing $\{u', v'\}$ would produce a better spanning tree, a contradiction. */

Minimum-Weight Spanning Trees (cont.)

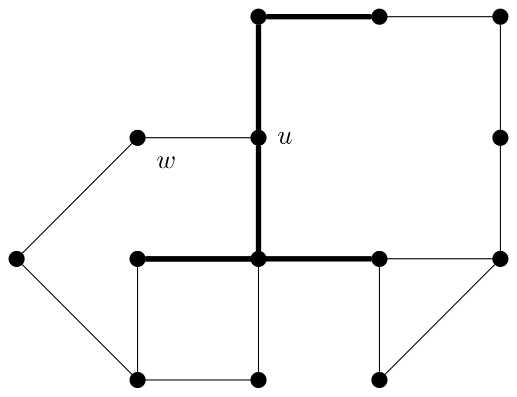


Figure: Finding the next edge of the MCST.

Source: redrawn from [Manber 1989, Figure 7.19].

Minimum-Weight Spanning Trees (cont.)

```
**Algorithm MST**($G$);
// A variant of Prim's algorithm
begin
    initially $T$ is the empty set;
    for all vertices $w$ do
        $w.mark := false$;  $w.cost := \infty$;
    let $(x, y)$ be a minimum cost edge in $G$;
    $x.mark := true$;
    for all edges $(x, z)$ do
        $z.edge := (x, z)$;  $z.cost := cost(x, z)$;
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected";  halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;  $z.cost := cost(w, z)$
end
```

Minimum-Weight Spanning Trees (cont.)

```
Algorithm Another_MST($G$);
// Prim's algorithm
begin
```

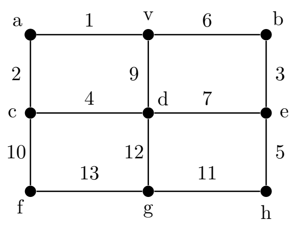
```
initially $T$ is the empty set;
for all vertices $w$ do
    $w.mark := false$;  $w.cost := \infty$;
$x.mark := true$; /* $x$ is an arbitrary vertex */
for all edges $(x, z)$ do
    $z.edge := (x, z)$;  $z.cost := cost(x, z)$;
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected";  halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;
                        $z.cost := cost(w, z)$
end
```

Time complexity: same as that of Dijkstra's algorithm.

Minimum-Weight Spanning Trees (cont.)



	v	a	b	c	d	e	f	g	h
v	-	v(1)	v(6)	∞	v(9)	∞	∞	∞	∞
a	-	-	v(6)	$a(2)$	v(9)	∞	∞	∞	∞
c	-	-	v(6)	-	c(4)	∞	$c(10)$	∞	∞
d	-	-	v(6)	-	-	d(7)	$c(10)$	d(12)	∞
b	-	-	-	-	-	b(3)	$c(10)$	d(12)	∞
e	-	-	-	-	-	-	$c(10)$	d(12)	e(5)
h	-	-	-	-	-	-	$c(10)$	h(11)	-
f	-	-	-	-	-	-	—	h(11)	-
g	-	-	-	-	-	-	—	-	-

Figure: An example of the minimum-cost spanning-tree algorithm. Source: redrawn from [Manber 1989, Figure 7.21].

8 All Shortest Paths

All Shortest Paths

Problem 15. *Given a weighted graph $G = (V, E)$ (directed or undirected) with nonnegative weights, find the minimum-length paths between all pairs of vertices.*

Floyd's Algorithm

```
**Algorithm All_Pairs_Shortest_Paths**($W$);
**begin**
    {initialization}
    **for** $i := 1$ to $n$ **do**
        **for** $j := 1$ to $n$ **do**
            **if** $(i, j) \in E$ **then** $W[i, j] := length(i, j)$
            **else** $W[i, j] := \infty$;
    **for** $i := 1$ to $n$ **do** $W[i, i] := 0$;

    **for** $m := 1$ to $n$ **do**  {the induction sequence}
        **for** $x := 1$ to $n$ **do**
            **for** $y := 1$ to $n$ **do**
                **if** $W[x, m] + W[m, y] < W[x, y]$ **then**
                    $W[x, y] := W[x, m] + W[m, y]$
**end**
```

/* The graph may contain edges with negative weights, as long as there is no cycle whose total weight is negative. */

Transitive Closure

Problem 16. *Given a directed graph $G = (V, E)$, find its transitive closure.*

```
**Algorithm Transitive_Closure**($A$);
**begin**
    {initialization omitted}
    **for** $m := 1$ to $n$ **do**
        **for** $x := 1$ to $n$ **do**
            **for** $y := 1$ to $n$ **do**
                **if** $A[x, m]$ and $A[m, y]$ **then**
                    $A[x, y] := true$
**end**
```

Transitive Closure (cont.)

```
**Algorithm Improved_Transitive_Closure**($A$);
**begin**
    {initialization omitted}
    **for** $m := 1$ to $n$ **do**
        **for** $x := 1$ to $n$ **do**
            **if** $A[x, m]$ **then**
                **for** $y := 1$ to $n$ **do**
                    **if** $A[m, y]$ **then**
                        $A[x, y] := true$
**end**
```