Homework 2

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Question1

Consider again the inductive definition in HW#1 for the set of all binary trees that store non-negative integer key values:

- The empty tree, denoted $\perp$, is a binary tree, storing no key value.
- If t_l and t_r are binary trees, then $node(k, t_l, t_r)$, where $k \in \mathbb{Z}$ and $k \geq 0$, is a also binary tree with the root storing key value k.

(a) Refine the definition to include only binary *search* trees where an inorder traversal of a binary search tree produces a list of all stored key values in *increasing* order.

(b) Further refine the definition to include only AVL trees, which are binary search trees where the heights of the left and the right children of every internal node differ by at most 1.

Question1

Inductive definition of all binary trees:

1. The empty tree, denoted $\perp$, is a binary tree.
2. If t_l and t_r are binary trees,
   then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$,
   is also a binary tree.

Question1(a)

Inductive definition of binary **search** trees:

1. The empty tree, denoted $\perp$, is a binary search tree.
2. If t_l and t_r are binary search trees, and $Max(t_l) \leq k \leq Min(t_r)$, then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$, is also a binary search tree.

$$Max(T) = \begin{cases} 0 & \text{if } T = \perp. \\ \mathbf{max}(k, \mathbf{max}(Max(t_l), Max(t_r))) & \text{if } T = node(k, t_l, t_r), \end{cases}$$

, where $\mathbf{max}(\mathrm{x}, \mathrm{y}) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise.} \end{cases}$

$$Min(T) = \begin{cases} +\infty & \text{if } T = \perp. \\ \mathbf{min}(k, \mathbf{min}(Min(t_l), Min(t_r))) & \text{if } T = node(k, t_l, t_r), \end{cases}$$

, where $\mathbf{min}(\mathrm{x}, \mathrm{y}) = \begin{cases} x & \text{if } x < y \\ y & \text{otherwise.} \end{cases}$

Question1(b)

Inductive definition of **AVL** binary search trees:

1. The empty tree, denoted $\perp$ (with a height of -1), is an AVL binary search tree.
2. If t_l and t_r are binary AVL binary search trees, and $Max(t_l) \leq k \leq Min(t_r)$ and $|H(t_r) - H(t_l)| \leq 1$, then $node(k, t_l, t_r)$, where $k \in Z$ and $k \geq 0$, is also an AVL binary search tree.

$$H(T) = \begin{cases} -1 & \text{if } T = \perp. \\ \mathbf{max}(H(t_r), H(t_l)) + 1 & \text{if } T = node(k, t_l, t_r), \end{cases}$$

, where $\mathbf{max}(\mathrm{x},\ \mathrm{y}) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise.} \end{cases}$

Question2

(2.30) A **full binary tree** is defined inductively as follows. (Note: some authors prefer using the name "perfect binary tree" or "complete binary tree", while reserving "full binary tree" for another variant of binary trees.) A full binary tree of height 0 consists of 1 node which is the root. A full binary tree of height $h+1$ consists of two full binary trees of height h whose roots are connected to a new root. Let T be a full binary tree of height h. The **height** of a node in T is h minus the node's distance from the root (e.g., the root has height h, whereas a leaf has height 0). Prove that the sum of the heights of all the nodes in T is $2^{h+1} - h - 2$.

Question2

[Claim] For a full binary tree T of height h, The sum of the heights of all nodes in T is $2^{h+1} - h - 2$.

[Base Case]$(h = 0)$ $2^1 - 0 - 2 = 0$

[Inductive Step] $(h > 0)$

[Inductive Hypothesis] For any full binary tree T of height $h - 1$, The sum of the heights of all nodes in T is $2^h - (h - 1) - 2$.

(2$\times$ sum of the heights of all nodes in T + height of root)

$= 2 \times (2^{(h-1)+1} - (h - 1) - 2) + h$

$= 2^{h+1} - 2h - 2 + h$

$= 2^{h+1} - h - 2$

Question3

(2.23) The **lattice** points in the plane are the points with integer coordinates. Let P be a polygon that does not cross itself (such a polygon is called **simple**) such that all of its vertices are lattice points (see Figure 1). Let p be the number of lattice points that are on the boundary of the polygon (including its vertices), and let q be the number of lattice points that are inside the polygon. Prove that the area of polygon is $\frac{p}{2} + q - 1$.

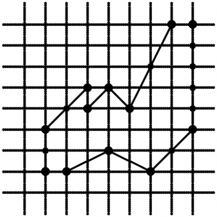
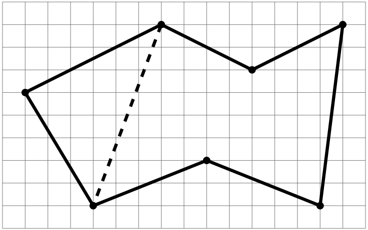


Figure 1: A simple polygon on the lattice points.

Question3

Any simple polygon with n vertices ($n \geq 4$) can be split into a triangle and another simple polygons with $n - 1$ vertices.
(Polygon Triangulation)



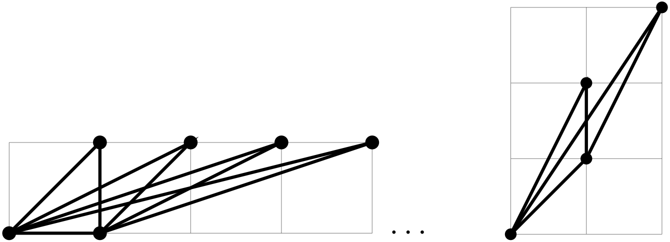
Question3

Prove by induction on the number n of vertices of the polygon.
[Base Case] $(n = 3)$

Prove by induction on the value of $p + q$:

[Base Case] $p + q = 3$ ($p = 3, q = 0$ is the only possibility).

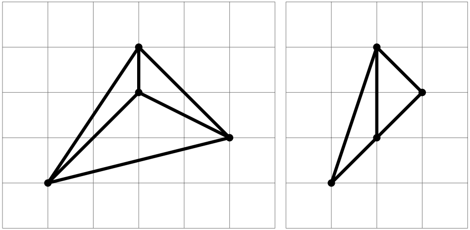
Area$=\frac{1}{2}=\frac{3}{2} + 0 - 1 = \frac{p}{2}$+q-1



[Inductive Step] $p + q > 3$

When $p + q > 3$, we can find a point v either inside this triangle or on one of its edges.

- [Case 1] If v is inside the triangle, connect v to three vertices, and split the large triangle into three small triangles.
- [Case 2] If v is on an edge, connect v to the opposite vertex, and split the large triangle into two small triangles.



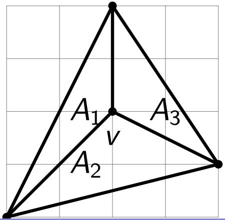
[Case 1] v is in the triangle

- Let s denote number of nodes on all inner lines.
- [Inductive Hypothesis]: area $= \frac{p}{2} + q - 1$ holds $\forall p + q \leq k$,
  Let p_i, q_i, A_i be the value of p, q and the area of a triangle A_i, $i \in \{1, 2, 3\}$ and p, q, A be the value of p, q and the area of a triangle A.
  - $p = p_1 + p_2 + p_3 - 2s + 2$
  - $q = q_1 + q_2 + q_3 + s - 3$
  - $A = A_1 + A_2 + A_3$

    $= (\frac{p_1}{2} + q_1 - 1) + (\frac{p_2}{2} + q_2 - 1) + (\frac{p_3}{2} + q_3 - 1)$

    $= \frac{p+2s-2}{2} + (q - s + 3) - 3$

    $= \frac{p}{2} + q - 1$



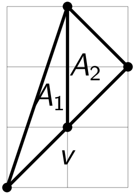
[Case 2] v is on the edge

- Let v denote number of nodes on the inner line.
- [Inductive Hypothesis]: area $= \frac{p}{2} + q - 1$ holds $\forall p + q \leq k$,
  Let p_i, q_i, A_i be the value of p, q and the area of a triangle A_i, $i \in \{1, 2\}$ and p, q, A be the value of p, q and the area of a triangle A.
  - $p = p_1 + p_2 - 2s + 2$
  - $q = q_1 + q_2 + s - 2$
  - $A = A_1 + A_2$

$$= (\frac{p_1}{2} + q_1 - 1) + (\frac{p_2}{2} + q_2 - 1)$$

$$= \frac{p+2s-2}{2} + (q - s + 2) - 2$$

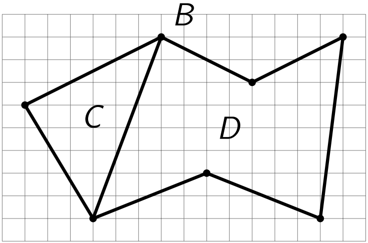
$$= \frac{p}{2} + q - 1$$



Question3

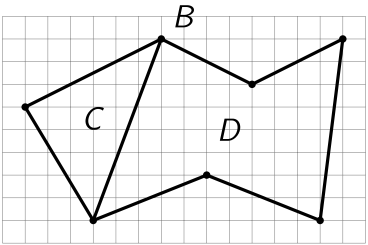
[Inductive step] ($n > 3$)

- Let B be a polygon with n vertices. B can be split into a triangle C and a polygon D with $n - 1$ vertices.
- [Inductive hypothesis]
  The area of a polygon with $n - 1$ vertices whose number of lattice points on the boundary of the polygon is p and the one inside the polygon is q is $\frac{p}{2} + q - 1$.



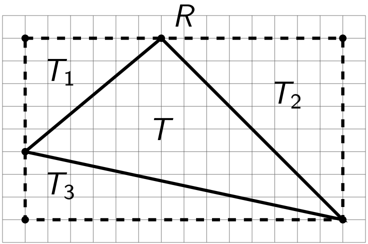
Question3

- Let s be the lattice points on the connected edge. We can find the relations below:
  - $p_B = p_C + p_D - 2s + 2$
  - $q_B = q_C + q_D + s - 2$
  - $A_B = A_C + A_D$
- $\frac{p_B}{2} + q_B - 1$
  $= \frac{p_C + p_D - 2s + 2}{2} + (q_C + q_D + s - 2) - 1$
  $= (\frac{p_C}{2} + q_C - 1) + (\frac{p_P}{2} + q_D - 1)$
  $= A_C + A_D = A_B$



Question3

Another proof of arbitrary triangle.



Arbitrary triangle = Rectangle - Right triangle*3:

1. Prove the hypothesis for arbitrary right triangles.
2. Prove the hypothesis for rectangles.
3. Prove the hypothesis for arbitrary triangles.

Question4

Let n be a natural number ($n \geq 0$) and p be a prime ($p \geq 2$). Let s be the sum of the p-ary digits in the representation of n in base p. Let m be the multiplicity of the factor p in $n!$, i.e., the maximum value m such that p^m divides $n!$. For example, if $n = 6$ and $p = 2$, then $n = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 110_2$ and hence $s = 1 + 1 + 0 = 2$. Moreover, 2^4 divides $6!$ but 2^5 does not and therefore $m = 4$.

Prove that

$$m = \frac{n - s}{p - 1} \ .$$

In the example above, $\frac{n-s}{p-1} = \frac{6-2}{2-1} = 4 = m$.

Question4

Rewrite $m = \frac{n-s}{p-1}$ as the function of n:

$$m(n!) = \frac{n - s(n)}{p - 1},$$

where $m(n!)$ is the maximum m such that p^m divides $n!$ and $s(n)$ is the sum of p-ary digits in the representation of n in base p.

[Claim]

For every $n \geq 0$ and any prime $p \geq 2$, $m(n!) = \frac{n-s(n)}{p-1}$ holds.

[Base Case] $(n = 0)$

For any prime p, $m(0!) = m(1) = 0$; $s(0) = 0$.

It's easy to know $m(0!) = \frac{n-s(n)}{p-1} = \frac{0-0}{p-1} = 0$ holds.

Question4

[Inductive Step] $(n > 0)$

Since p is prime, we have the relation
$m(n!) = m((n-1)!) + m(n)$ derived from $k! = k(k-1)!$

$$P^m | n(n-1)!$$
$$P^{m'} | (n-1)! \rightarrow m = m' + \frac{n}{p^m}$$

We also have the relation

$$s(n) = s(n-1) + 1 - (p-1)m(n),$$

where $m(n)$ can express the number of $p-1$ digits at the end of $n-1$ represented in base p.

For example, if $n = 27$ and $p = 3$, then
$n = 1 \times 3^3 + 0 \times 3^2 + 0 \times 0^1 + 0 \times 3^0 = 1000_3$ and $n - 1 = 26 = 222_3$.
$s(27) = s(26) + 1 - (3-1)m(27)$, where $m(27) = 3$.

Question4

[Inductive Step] ($n > 0$)

[Inductive hypothesis] By observation above, for every $n > 0$ and any prime p, the following equations holds.

$$\begin{cases} m(n!) = m((n-1)!) + m(n) \\ s(n) = s(n-1) + 1 - (p-1)m(n) \\ m(n!) = \dfrac{n - s(n)}{p-1} \end{cases}$$

$$\begin{aligned} m(n!) = \frac{n - s(n)}{p-1} &= \frac{n - [s(n-1) + 1 - (p-1)m(n)]}{p-1} \\ &= \frac{n - [s(n-1) + 1 - (p-1)(m(n!) - m((n-1)!)]}{p-1} \\ &= \frac{(n-1) - s(n-1)}{p-1} + m(n!) - m((n-1)!) \\ &\overset{I.H.}{=} m((n-1)!) + m(n!) - m((n-1)!) = m(n!) \end{aligned}$$

Question5

Consider the following algorithm for computing the square of the input number n, which is assumed to be a positive integer.

Algorithm mySquare(n);
begin
    // assume that $n > 0$
    $x := n$;
    $y := 0$;
    while $x > 0$ **do**
        $y := y + 2 \times x - 1$;
        $x := x - 1$
    od;
    mySquare $:= y$
end

State a suitable loop invariant for the while loop and prove its correctness. The loop invariant should be strong enough for deducing that, when the while loop terminates, the value of y equals the square of n.

Question5: loop invariant

A suitable loop *Inv* invariant need to meet the following two conditions:

- [Initial condition]
  Inv needs to hold before the loop starts.
- [Each iteration in loop]
  Inv needs to hold in each iteration if the iteration need to be processed.

Question5: loop invariant

Observe:

State a loop invariant:

$x := n;$
$y := 0;$
while $x > 0$ **do**
  $y := y + 2 \times x - 1;$
  $x := x - 1;$
end while
$mySquare := y;$

List x and y:

x	n	$n-1$	$n-2$	$n-3$	$\cdots$	$n-k$	$\cdots$
y	0	$2n-1$	$4n-4$	$6n-9$	$\cdots$	$(2k)n-k^2$	$\cdots$

Remember, this is not a proof!
It's just a way to find out the loop invariant.

Question5: loop invariant

Now we have $x = n - k, y = (2k)n - k^2$.

$$\overset{k=n-x}{\Rightarrow} y = 2(n-x)n - (n-x)^2$$
$$= 2n^2 - 2xn - (n^2 - 2xn + x^2)$$
$$= n^2 - x^2$$

We find the loop invariant are $\{0 \le x \le n \wedge y = n^2 - x^2\}$.

Check whether the loop invariant can ensure that $mySquare = n^2$ at the of loop.
When the while loop terminates, the value of x is 0.
The loop invariant ensures: $mySquare = y = n^2 - x^2 = n^2 - 0^2 = n^2$.

Question5: prove the correctness

[Claim] Loop invariant $\{0 \le x \le n \wedge y = n^2 - x^2\}$.

[Base case] 0-th iteration: $\{0 \le n \le n \wedge y = n^2 - n^2 = 0\}$.

[Inductive step] k-th iteration

[Induction hypothesis] After $(k-1)$-th iteration, loop invariant holds.

Let (x', y', n') be the new (x, y, n) values at the next iteration of the while loop (while condition be satisfied).

After one iteration, we have $\begin{cases} y' = y + 2x - 1, \\ x' = x - 1, \\ n' = n \end{cases}$

Question5: prove the correctness

To prove

- $0 \leq x' \leq n'$ (is automatically true, x' must ≥ 1 because of the while condition and $n' = n$).
- $y' = n'^2 - x'^2$.

$$y' = y + 2x - 1$$
$$\overset{I.H.}{\Rightarrow} y' = n^2 - x^2 + 2x - 1$$
$$= n'^2 - (x'+1)^2 + 2(x'+1) - 1$$
$$= n'^2 - x'^2 - 2x' - 1 + 2x' + 2 - 1$$
$$= n'^2 - x'^2$$