Algorithms 2023: Data Structures

A Supplement (Based on [Manber 1989])

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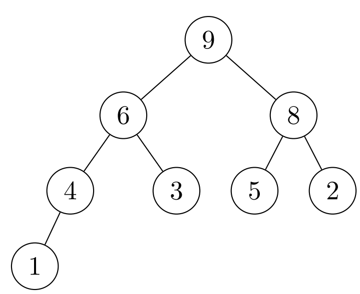
1 Heaps

Heaps

- A (max binary) heap is a complete binary tree whose keys satisfy the heap property: *the key of every node is greater than or equal to the key of any of its children*.
- It supports the two basic operations of a priority queue:
  - *Insert*(x): insert the key x into the heap.
  - *Remove*(): remove and return the largest key from the heap.

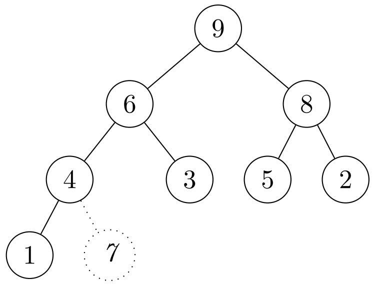
Heaps (cont.)

- A complete binary tree can be represented implicitly by an array A as follows:
  1. The root is stored in $A[1]$.
  2. The left child of $A[i]$ is stored in $A[2i]$ and the right child is stored in $A[2i+1]$.

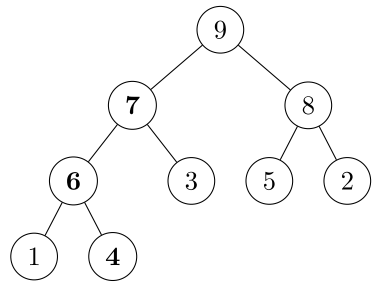


1	2	3	4	5	6	7	8	9	10	11	12
9	6	8	4	3	5	2	1	_	_	_	_

Heaps (cont.)



Before *Insert*(7)

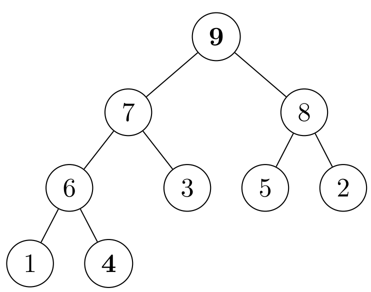


After *Insert*(7)

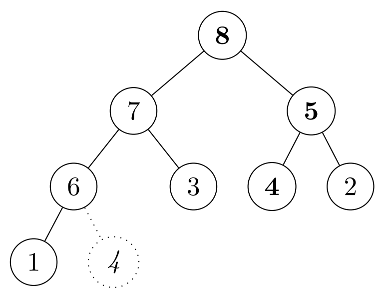
Heaps (cont.)

```
Algorithm Insert_to_Heap (A, n, x);
begin
        n := n + 1;
        A[n] := x;
        child := n;
        parent := n div 2;
        while parent ≥ 1 do
              if A[parent] < A[child] then
                 swap(A[parent], A[child]);
                 child := parent;
                 parent := parent div 2
              else parent := 0
end
```

Heaps (cont.)



Before *Remove*()



After *Remove*()

Heaps (cont.)

```
Algorithm Remove_Max_from_Heap (A, n);
begin
    if n = 0 then print "the heap is empty"
    else Top_of_the_Heap := A[1];
        A[1] := A[n]; n := n − 1;
        parent := 1; child := 2;
        while child ≤ n − 1 do
              if A[child] < A[child + 1] then
                 child := child + 1;
```

if $A[child] > A[parent]$ **then**
$swap(A[parent], A[child])$;
$parent := child$;
$child := 2 * child$
else $child := n$
end

2 AVL Trees

AVL Trees

Definition 1. An AVL tree is a binary search tree such that, for every node, the difference between the heights of its left and right subtrees is at most 1 (the height of an empty tree is defined as 0).

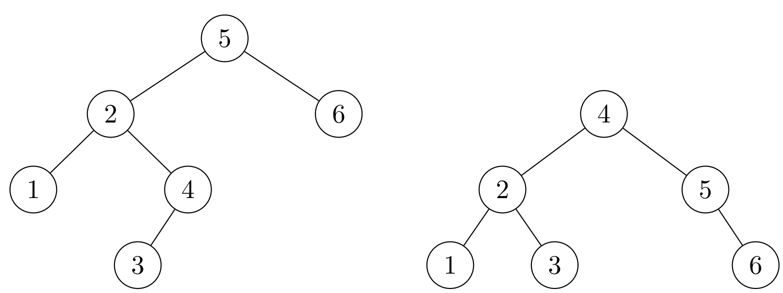
This definition guarantees a maximal height of $O(\log n)$ for any AVL tree of n nodes.

/* Let $G(h)$ denote the least possible number of nodes contained in an AVL tree of height h; the empty tree has height -1 and a single-node tree has height 0. A recurrence relation for $G(h)$ can be defined as follows:

$$\begin{cases} G(-1) & = 0 \\ G(0) & = 1 \\ G(h) & = G(h-1) + G(h-2) + 1, \quad h \geq 1 \end{cases}$$

A precise solution to $G(h)$ may be derived by establishing the relation $G(h) = F(h+3) - 1$, where $F(i)$ is the i-th Fibonacci number (as defined in Chapter 3.5 of Manber's book) for which we already know the closed form; the proof is quite simple by induction. So, for any AVL tree with n nodes and of height h, $n \geq G(h) \geq F(h+3) - 1 \geq ca^h$ (for some positive constants c and a and sufficiently large n). It follows that $h = O(\log n)$. */

AVL Trees (cont.)



A binary search tree but NOT an AVL tree

A binary search tree and also an AVL tree

AVL Trees (cont.)

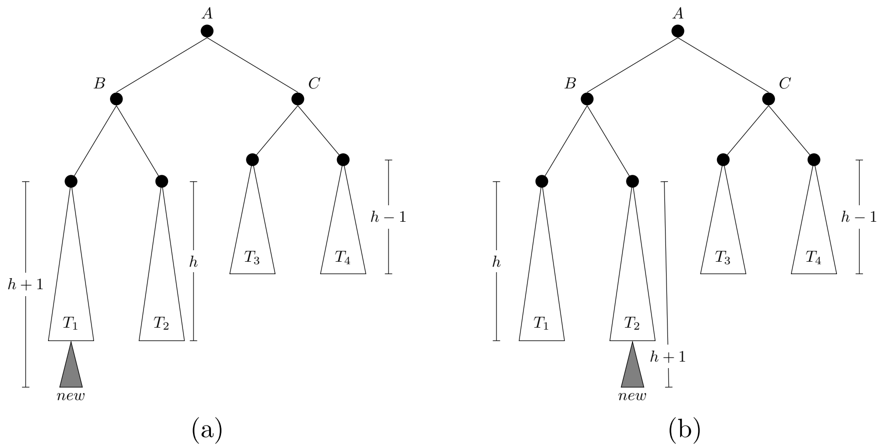


Figure: Insertions that invalidate the AVL property. Note that this tree rooted at A shown here may be part of a larger AVL tree.

Source: redrawn from [Manber 1989, Figure 4.13].

AVL Trees (cont.)

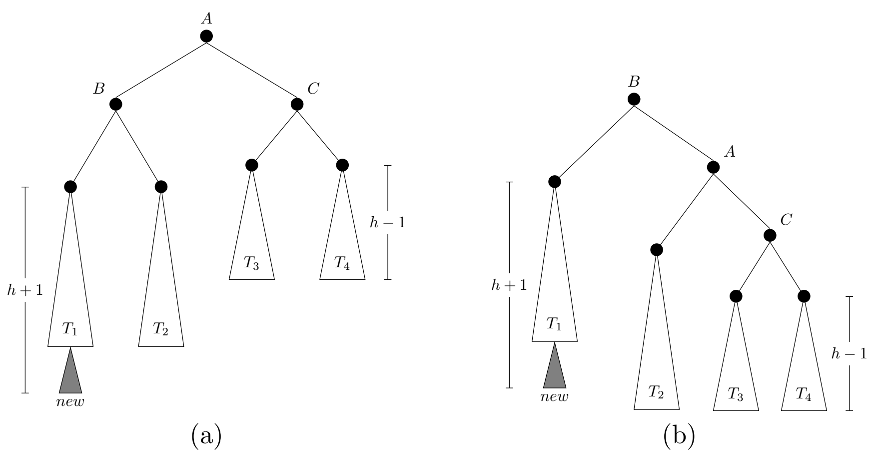


Figure: A single rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.14].

AVL Trees (cont.)

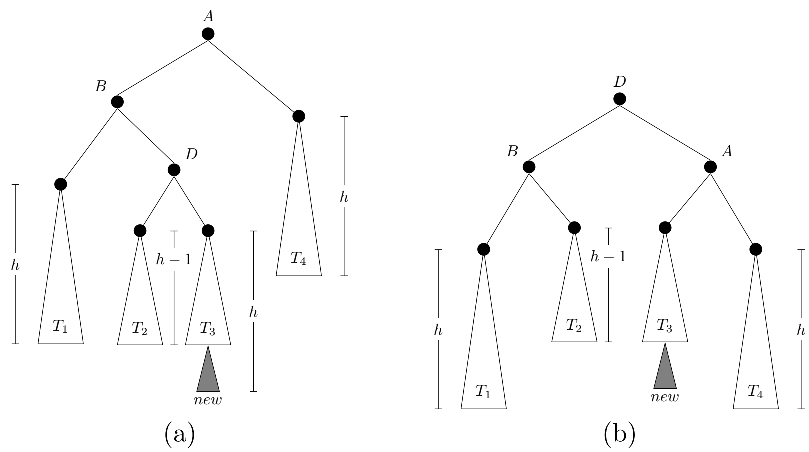


Figure: A double rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.15].

3 Union-Find

Union-Find

- There are n elements $x_1, x_2, \cdots, x_n$ divided into groups. Initially, each element is in a group by itself.
- Two operations on the elements and groups:
  - $find(A)$: returns the name of A's group.
  - $union(A, B)$: combines A's and B's groups to form a new group with a unique name.
- To tell if two elements are in the same group, one may issue a find operation for each element and see if the returned names are the same.

Union-Find (cont.)

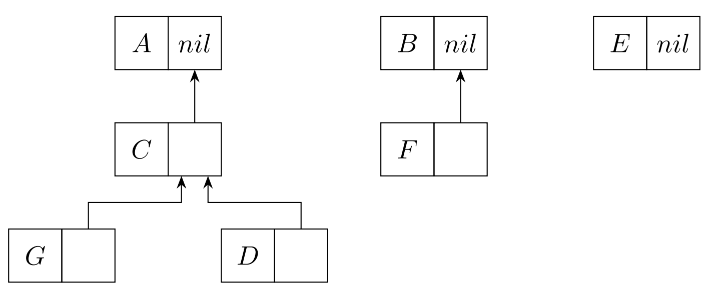


Figure: The representation for the union-find problem.

Source: redrawn from [Manber 1989, Figure 4.16].

Balancing

- The root also stores the number of elements in (i.e., the size of) its group.
- To *balance* the tree resulted from a union operation, *let the smaller group join the larger group* and update the size of the larger group accordingly.

Theorem 2 (Theorem 4.2)**.** *If balancing is used, then any tree of height h (≥ 0) must contain at least 2^h elements.*

/* This can be proven by induction on the number n (≥ 1) of elements/nodes. */

- Any sequence of m find or union operations (where $m \geq n$) takes $O(m \log n)$ steps.

Union-Find (cont.)

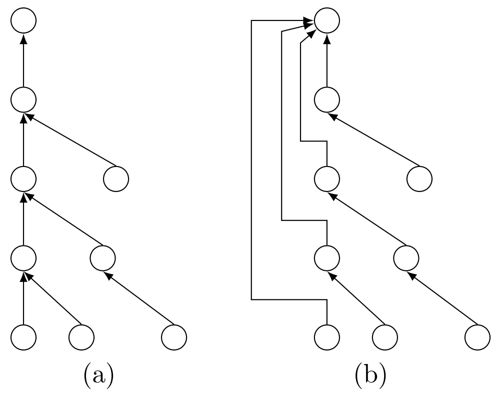


Figure: Path compression: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.17].

Effect of Path Compression

Theorem 3 (Theorem 4.3)**.** *If both balancing and path compression are used, any sequence of m find or union operations (where $m \geq n$) takes $O(m \log^* n)$ steps.*

The value of $\log^* n$ intuitively equals the number of times that one has to apply log to n to bring its value down to 1.

/* When $n = 2^{2^{2^2}}$, $\log n = 2^{2^2} = 16$ and $\log^* n = 4$. When $n = 2^{2^{2^{2^2}}}$, $\log n = 2^{2^{2^2}} = 2^{16} = 65536$ and $\log^* n = 5$. */

Code for Union-Find

```
Algorithm Union_Find_Init(A,n);
begin
  for i := 1 to n do
      A[i].parent := nil;
      A[i].size := 1
end

Algorithm Find(a);
begin
  if A[a].parent <> nil then
     A[a].parent := Find(A[a].parent);
     Find := A[a].parent;
  else
     Find := a
end
```

Code for Union-Find (cont.)

```
Algorithm Union(a,b);
begin
  x := Find(a);
  y := Find(b);
  if x <> y then
     if A[x].size > A[y].size then
        A[y].parent := x;
        A[x].size := A[x].size + A[y].size;
     else
        A[x].parent := y;
        A[y].size := A[y].size + A[x].size
end
```