Data Structures

A Supplement
(Based on [Manber 1989])

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Heaps

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  the key of every node is greater than or equal to the key of any of its children.
- It supports the two basic operations of a priority queue:

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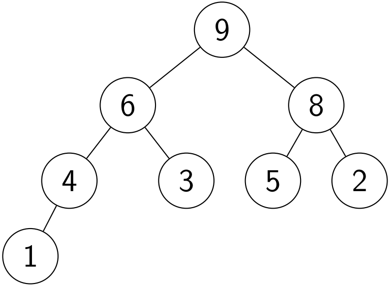
  the key of every node is greater than or equal to the key of any of its children.
- It supports the two basic operations of a priority queue:
  - *Insert*(x): insert the key x into the heap.
  - *Remove*(): remove and return the largest key from the heap.

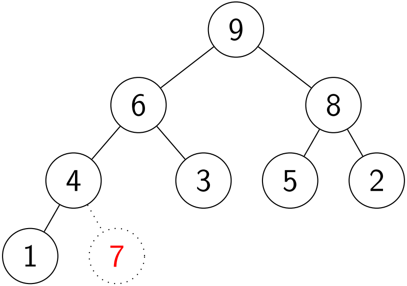
Heaps (cont.)

- A complete binary tree can be represented implicitly by an array A as follows:
  1. The root is stored in $A[1]$.
  2. The left child of $A[i]$ is stored in $A[2i]$ and the right child is stored in $A[2i+1]$.

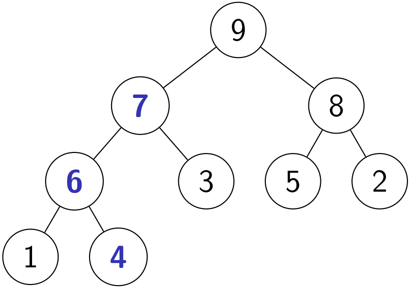


1	2	3	4	5	6	7	8	9	10	11	12
9	6	8	4	3	5	2	1	_	_	_	_

Heads (cont.)



Before *Insert*(7)



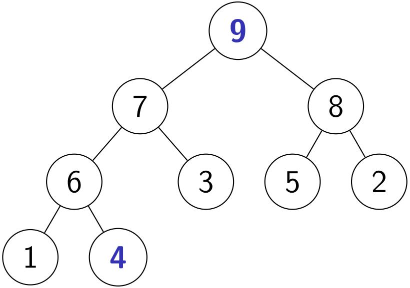
After *Insert*(7)

Heaps (cont.)

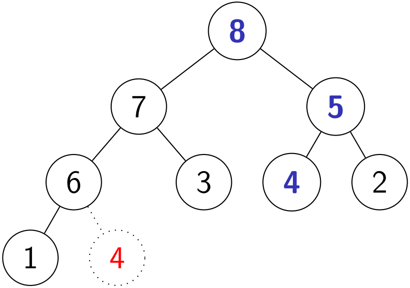
Algorithm Insert_to_Heap (A, n, x);
begin

$n := n + 1$;
$A[n] := x$;
$child := n$;
$parent := n \; div \; 2$;
while $parent \geq 1$ **do**
  if $A[parent] < A[child]$ **then**
    $swap(A[parent], A[child])$;
    $child := parent$;
    $parent := parent \; div \; 2$
  else $parent := 0$
end

Heaps (cont.)



Before *Remove*()



After *Remove*()

Heaps (cont.)

Algorithm Remove_Max_from_Heap (A, n);
begin
    if $n = 0$ **then** print "the heap is empty"
    else $Top\_of\_the\_Heap := A[1]$;
        $A[1] := A[n]$; $n := n - 1$;
        $parent := 1$; $child := 2$;
        while $child \leq n - 1$ **do**
            if $A[child] < A[child + 1]$ **then**
                $child := child + 1$;
            if $A[child] > A[parent]$ **then**
                $swap(A[parent], A[child])$;
                $parent := child$;
                $child := 2 * child$
            else $child := n$
end

AVL Trees

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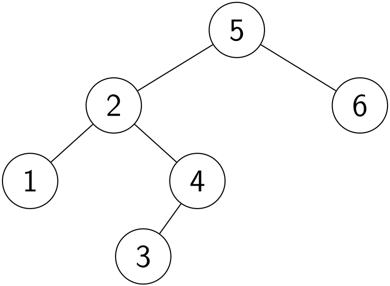
Definition

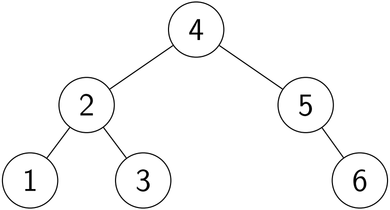
An AVL tree is a binary search tree such that, for every node, the difference between the heights of its left and right subtrees is at most 1 (the height of an empty tree is defined as 0).

This definition guarantees a maximal height of $O(\log n)$ for any AVL tree of n nodes.

AVL Trees (cont.)



A binary search tree
but NOT an AVL tree



A binary search tree
and also an AVL tree

AVL Trees (cont.)

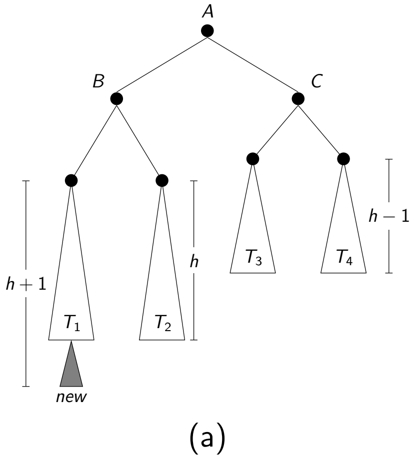


Figure: Insertions that invalidate the AVL property. Note that this tree rooted at A shown here may be part of a larger AVL tree.

Source: redrawn from [Manber 1989, Figure 4.13].

AVL Trees (cont.)

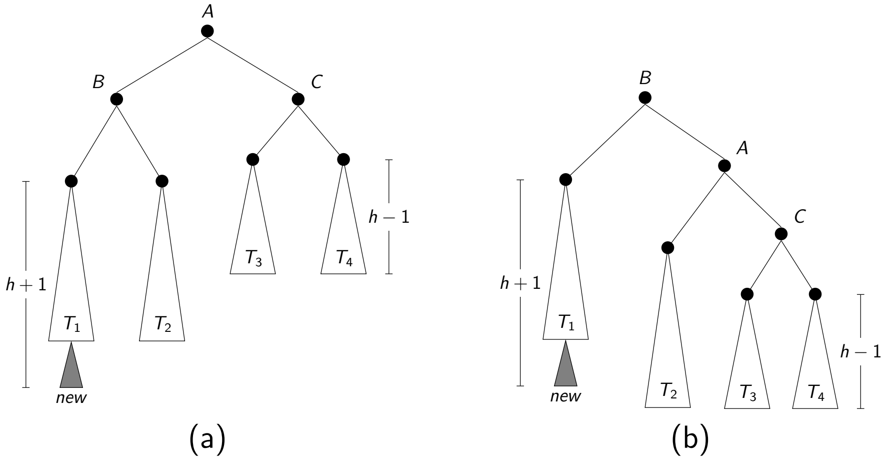


Figure: A single rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.14].

AVL Trees (cont.)

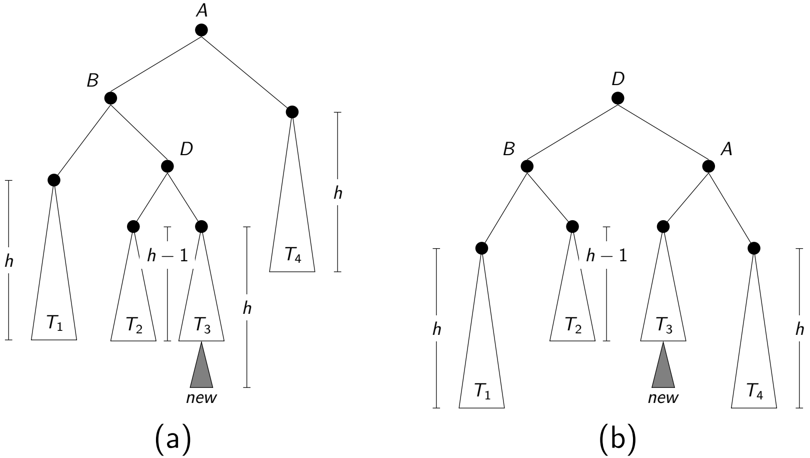


Figure: A double rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.15].

Union-Find

- There are n elements $x_1, x_2, \cdots, x_n$ divided into groups. Initially, each element is in a group by itself.
- Two operations on the elements and groups:
  - *find*(A): returns the name of A's group.
  - *union*(A, B): combines A's and B's groups to form a new group with a unique name.
- To tell if two elements are in the same group, one may issue a find operation for each element and see if the returned names are the same.

Union-Find (cont.)

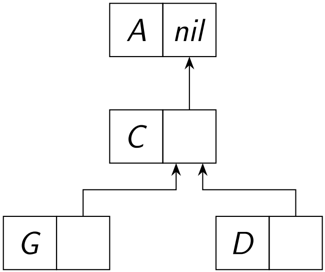


Figure: The representation for the union-find problem.

Source: redrawn from [Manber 1989, Figure 4.16].

Balancing

- The root also stores the number of elements in (i.e., the size of) its group.
- To *balance* the tree resulted from a union operation, *let the smaller group join the larger group* and update the size of the larger group accordingly.

Theorem (Theorem 4.2)

If balancing is used, then any tree of height h (≥ 0) must contain at least 2^h elements.

- Any sequence of m find or union operations (where $m \geq n$) takes $O(m \log n)$ steps.

Union-Find (cont.)

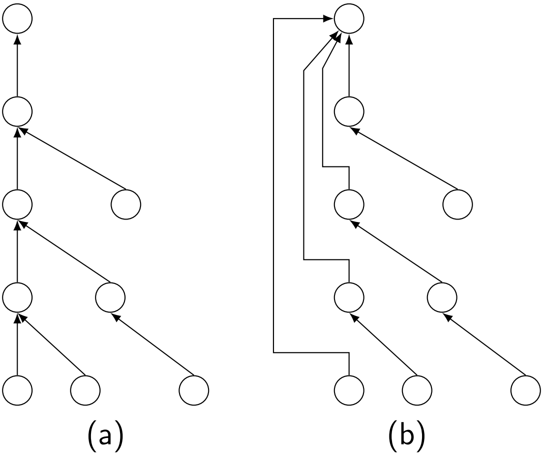


Figure: Path compression: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.17].

Effect of Path Compression

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Theorem (Theorem 4.3)

If both balancing and path compression are used, any sequence of m find or union operations (where $m \geq n$) takes $O(m \log^* n)$ steps.

The value of $\log^* n$ intuitively equals the number of times that one has to apply log to n to bring its value down to 1.

Code for Union-Find

```
Algorithm Union_Find_Init(A,n);
begin
  for i := 1 to n do
      A[i].parent := nil;
      A[i].size := 1
end

Algorithm Find(a);
begin
  if A[a].parent <> nil then
     A[a].parent := Find(A[a].parent);
     Find := A[a].parent;
  else
     Find := a
end
```

Code for Union-Find (cont.)

```
Algorithm Union(a,b);
begin
  x := Find(a);
  y := Find(b);
  if x <> y then
     if A[x].size > A[y].size then
        A[y].parent := x;
        A[x].size := A[x].size + A[y].size;
     else
        A[x].parent := y;
        A[y].size := A[y].size + A[x].size
end
```