Algorithms 2023: Design by Induction

(Based on [Manber 1989])

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1 Introduction

Introduction

- It is not necessary to design the steps required to solve a problem from scratch.
- It is sufficient to guarantee the following:
  1. It is possible to solve one small instance or a few small instances of the problem. (base case)
  2. A solution to every problem/instance can be constructed from solutions to smaller problems/instances. (inductive step)

2 Evaluating Polynomials

Evaluating Polynomials

Problem 1. *Given a sequence of real numbers a_n, a_{n-1}, $\cdots$, a_1, a_0, and a real number x, compute the value of the polynomial*

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

Evaluating Polynomials (cont.)

- Let $P_{n-1}(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.
- **Induction hypothesis** (first attempt)

  We know how to evaluate a polynomial represented by the input a_{n-1}, $\cdots$, a_1, a_0, at the point x, i.e., we know how to compute $P_{n-1}(x)$.
- $P_n(x) = a_n x^n + P_{n-1}(x)$.
- Number of multiplications:
$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}.$$

Evaluating Polynomials (cont.)

- **Induction hypothesis** (second attempt)

  We know how to compute $P_{n-1}(x)$, and we know how to compute x^{n-1}.

- $P_n(x) = a_n x(x^{n-1}) + P_{n-1}(x)$.

- Number of multiplications: $2n - 1$.

Evaluating Polynomials (cont.)

- Let $P'_{n-1}(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$.

- **Induction hypothesis** (final attempt)

  We know how to evaluate a polynomial represented by the coefficients a_n, a_{n-1}, $\cdots$, a_1, at the point x, i.e., we know how to compute $P'_{n-1}(x)$.

- $P_n(x) = P'_n(x) = P'_{n-1}(x) \cdot x + a_0$.

Evaluating Polynomials (cont.)

- More generally,

$$\begin{cases} P'_0(x) = a_n \\ P'_i(x) = P'_{i-1}(x) \cdot x + a_{n-i}, \text{ for } 1 \leq i \leq n \end{cases}$$

- Number of multiplications: n.

Evaluating Polynomials (cont.)

Algorithm Polynomial_Evaluation $(\bar{a}, x)$;
begin
  $P := a_n$;
  for $i := 1$ **to** n **do**
    $P := x * P + a_{n-i}$
end

This algorithm is known as *Horner's rule*.

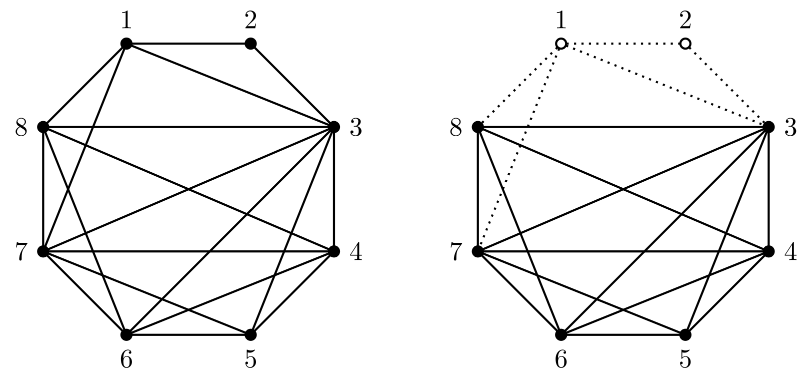
3 Maximal Induced Subgraph

Maximal Induced Subgraph

Problem 2. *Given an undirected graph $G = (V, E)$ and an integer k, find an induced subgraph $H = (U, F)$ of G of maximum size such that all vertices of H have degree $\geq k$ (in H), or conclude that no such induced subgraph exists.*

Design Idea: in the inductive step, we try to remove one vertex (that cannot possibly be part of the solution) to get a smaller instance.

Maximal Induced Subgraph (cont.)



A graph G of eight nodes.

Maximal induced subgraph of G when $k = 4$.

Maximal Induced Subgraph (cont.)

- Recursive:

  Algorithm Max_Ind_Subgraph (G, k);
  begin
      if the degree of every vertex of $G \geq k$ **then**
        Max_Ind_Subgraph := G;
      else let v be a vertex of G with degree $< k$;
          Max_Ind_Subgraph := Max_Ind_Subgraph($G - v, k$);
  end

  /* $G - v$ denotes the graph obtained from G by removing vertex v and every edge incident to v. */

- Iterative:

  Algorithm Max_Ind_Subgraph (G, k);
  begin
      while the degree of some vertex v of $G < k$ **do**
          $G := G - v$;
      Max_Ind_Subgraph := G;
  end

4 One-to-One Mapping

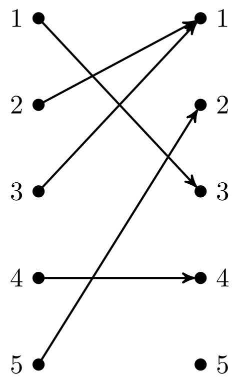
One-to-One Mapping

Problem 3. *Given a finite set A and a mapping f from A to itself, find a subset $S \subseteq A$ with the maximum number of elements, such that (1) the function f maps every element of S to another element of S (i.e., f maps S into itself), and (2) no two elements of S are mapped to the same element (i.e., f is one-to-one when restricted to S).*

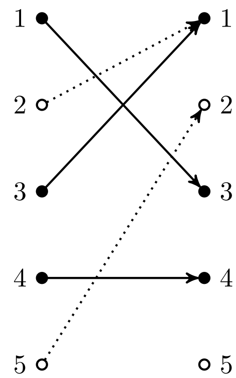
Design Idea: similar to the previous problem; in the inductive step, we try to remove one element (that cannot possibly be part of the solution) to get a smaller instance.

An element that is not mapped to may be removed.

One-to-One Mapping (cont.)



A given set A and
a mapping to itself.



The maximal selected subset S and
the remaining 1-to-1 mapping.

One-to-One Mapping (cont.)

Algorithm Mapping (f, n);
begin
  $S := A$;
  for $j := 1$ **to** n **do** $c[j] := 0$;
  for $j := 1$ **to** n **do** increment $c[f[j]]$;
  for $j := 1$ **to** n **do**
    if $c[j] = 0$ **then** put j in Queue;
  while Queue not empty **do**
    remove i from the top of Queue;
    $S := S - \{i\}$;
    decrement $c[f[i]]$;
    if $c[f[i]] = 0$ **then** put $f[i]$ in Queue
end

5 Celebrity

Celebrity

Problem 4. *Given an $n \times n$ adjacency matrix, determine whether there exists an i (the "celebrity") such that all the entries in the i-th column (except for the ii-th entry) are 1, and all the entries in the i-th row (except for the ii-th entry) are 0.*

/* In an adjacency matrix representing a directed graph, a 1 in the i-th row and the j-th column indicates that there is a directed edge from node i to node j (or i knows j), and a 0 indicates otherwise. */

Note: A celebrity corresponds to a sink of the directed graph.

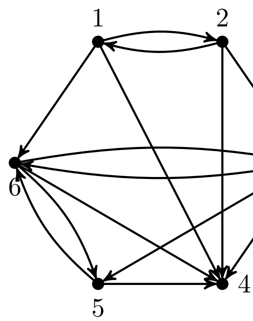
Note: Every directed graph has at most one sink.

/* Proof by contradiction. */

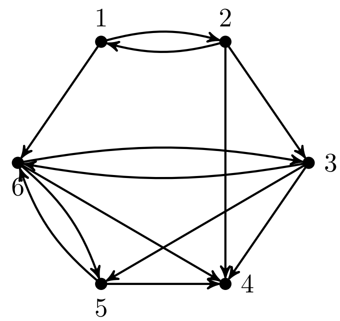
Motivation: the trivial solution has a time complexity of $O(n^2)$. Can we do better, in $O(n)$?

To achieve $O(n)$ time, we must reduce the problem size by at least one in constant time.

Celebrity (cont.)



A graph of six nodes with a sink (node 4).



A graph of six nodes without a sink.

Celebrity (cont.)

Basic idea: check whether i knows j.

In either case, one of the two may be eliminated.

/* If i knows j, then i is not a celebrity. If i does not know j, then j is not a celebrity. */

The $O(n)$ algorithm proceeds in two stages:

- Eliminate a node every round until only one is left.

  /* The node that remains is not necessarily a celebrity, as we have not checked whether it knows any previously deleted node or the other way around. */

- Check whether the remaining one is truly a celebrity.

Celebrity (cont.)

Algorithm Celebrity (*Know*);
begin
    $i := 1$;
    $j := 2$;
    $next := 3$;
    while $next \le n + 1$ **do**
        if $Know[i, j]$ **then** $i := next$
                    else $j := next$;
        $next := next + 1$;
    if $i = n + 1$ **then** $candidate := j$
            else $candidate := i$;

Celebrity (cont.)

    $wrong := false$;
    $k := 1$;
    $Know[candidate, candidate] := false$;
    while *not wrong and* $k \le n$ **do**
        if $Know[candidate, k]$ **then** $wrong := true$;

if *not Know*$[k, candidate]$ **then**
if $candidate \neq k$ **then** $wrong := true$;
$k := k + 1$;
if *not wrong* **then** $celebrity := candidate$
else $celebrity := 0$;
end

6 The Skyline Problem

The Skyline Problem

Problem 5. *Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimension) of these buildings, eliminating hidden lines.*

Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

Compare: adding buildings one by one to an existing skyline vs. merging two skylines of about the same size

The Skyline Problem

- Adding one building at a time:
$$\begin{cases} T(1) = O(1) \\ T(n) = T(n-1) + O(n), n \geq 2 \end{cases}$$
Time complexity: $O(n^2)$.

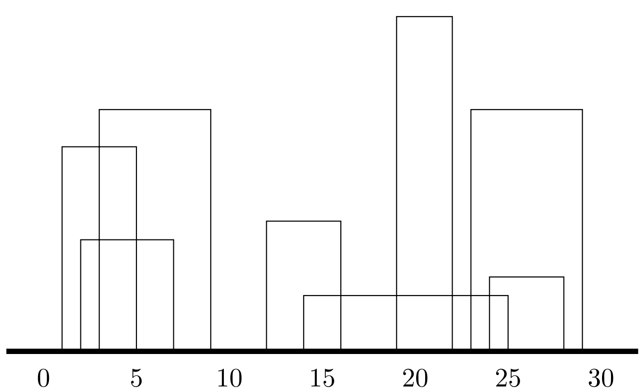
/* $T(n) = T(n-1) + O(n) = (T(n-2) + O(n-1)) + O(n) = \cdots = O(1) + O(2) + \cdots + O(n) = O(n^2)$. */

- Merging two skylines every round:
$$\begin{cases} T(1) = O(1) \\ T(n) = 2T(\frac{n}{2}) + O(n), n \geq 2 \end{cases}$$
Time complexity: $O(n \log n)$.

/* Apply the master theorem. Here, $a = 2$, $b = 2$, $k = 1$, and $b^k = 2 = a$. */

Representation of a Skyline

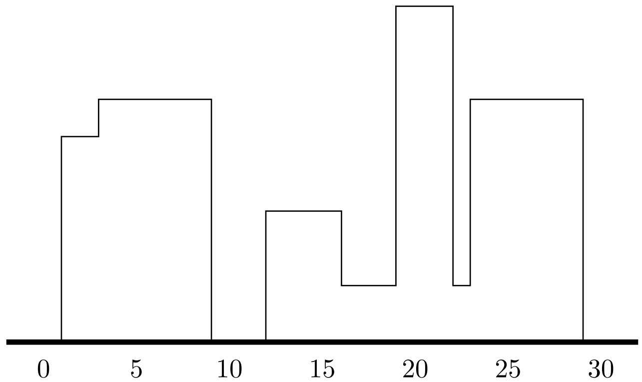
Input: (1,**11**,5), (2,**6**,7), (3,**13**,9), (12,**7**,16), (14,**3**,25), (19,**18**,22), (23,**13**,29), and (24,**4**,28).



Source: adapted from [Manber 1989, Figure 5.5(a)].

Representation of a Skyline (cont.)

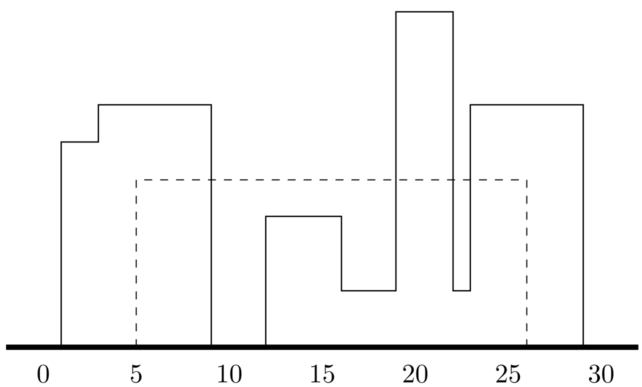
Representation: (1,**11**,3,**13**,9,**0**,12,**7**,16,**3**,19,**18**,22,**3**,23,**13**,29).



Source: adapted from [Manber 1989, Figure 5.5(b)].

Adding a Building

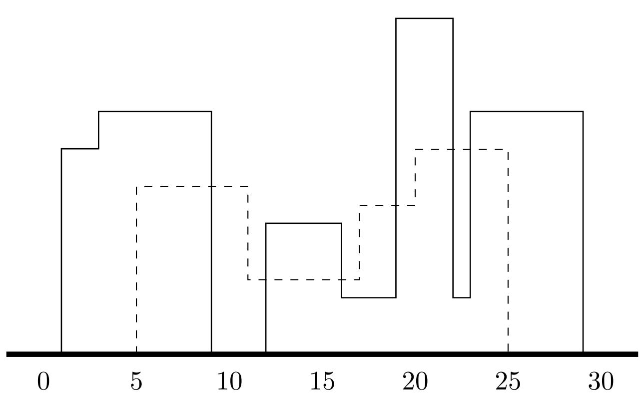
- Add (5,**9**,26) to (1,**11**,3,**13**,9,**0**,12,**7**,16,**3**,19,**18**,22,**3**,23,**13**,29).



Source: adapted from [Manber 1989, Figure 5.6].

- The skyline becomes (1,**11**,3,**13**,9,**9**,19,**18**,22,**9**,23,**13**,29).

Merging Two Skylines



Source: adapted from [Manber 1989, Figure 5.7].

7 Balance Factors in Binary Trees

Balance Factors in Binary Trees

Problem 6. *Given a binary tree T with n nodes, compute the balance factors of all nodes.*

The balance factor of a node is defined as the difference between the height of the node's left subtree and the height of the node's right subtree.

Motivation: an example of why we must strengthen the hypothesis (and hence the problem to be solved).

Balance Factors in Binary Trees (cont.)

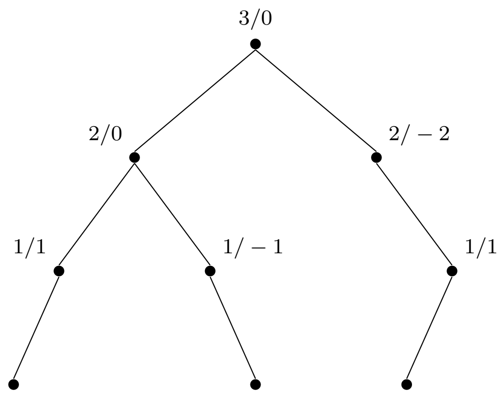


Figure: A binary tree. The numbers represent h/b, where h is the height and b is the balance factor.
Source: redrawn from [Manber 1989, Figure 5.8].

Balance Factors in Binary Trees (cont.)

- **Induction hypothesis**

  We know how to compute balance factors of all nodes in trees that have $< n$ nodes.

- **Stronger induction hypothesis**

  We know how to compute balance factors and heights of all nodes in trees that have $< n$ nodes.

8 Maximum Consecutive Subsequence

Maximum Consecutive Subsequence

Problem 7. *Given a sequence x_1, x_2, $\cdots$, x_n of real numbers (not necessarily positive), find a subsequence x_i, x_{i+1}, $\cdots$, x_j (of consecutive elements) such that the sum of the numbers in it is maximum over all subsequences of consecutive elements.*

Example: In the sequence $(2, -3, 1.5, -1, 3, -2, -3, 3)$, the maximum subsequence is $(1.5, -1, 3)$.

Motivation: another example of strengthening the hypothesis.

Maximum Consecutive Subsequence (cont.)

- **Induction hypothesis**

  We know how to find the maximum subsequence in sequences of size $< n$.

- **Stronger induction hypothesis**

  We know how to find, in sequences of size $< n$, the maximum subsequence overall and the maximum subsequence that is a suffix.

  Reasoning: the maximum subsequence of problem size n is obtained either

  - directly from the maximum subsequence of problem size $n - 1$ or
  - from appending the n-th element to the maximum suffix of problem size $n - 1$.

Maximum Consecutive Subsequence (cont.)

Algorithm Max_Consec_Subseq (X, n);
begin
    Global_Max := 0;
    Suffix_Max := 0;
    for $i := 1$ **to** n **do**
        if $x[i]$ + *Suffix_Max* > *Global_Max* **then**
            Suffix_Max := *Suffix_Max* + $x[i]$;
            Global_Max := *Suffix_Max*
        else if $x[i]$ + *Suffix_Max* > 0 **then**
                Suffix_Max := *Suffix_Max* + $x[i]$
        else *Suffix_Max* := 0
end

9 The Knapsack Problem

The Knapsack Problem

Problem 8. *Given an integer K and n items of different sizes such that the i-th item has an integer size k_i, find a subset of the items whose sizes sum to exactly K, or determine that no such subset exists.*

Design Idea: use strong induction so that solutions to all smaller instances may be used.

The Knapsack Problem (cont.)

- Let $P(n, K)$ denote the problem where n is the number of items and K is the size of the knapsack.

- **Induction hypothesis**

  We know how to solve $P(n - 1, K)$.

- **Stronger induction hypothesis**

  We know how to solve $P(n - 1, k)$, for all $0 \leq k \leq K$.

  Reasoning: $P(n, K)$ has a solution if either

  - $P(n - 1, K)$ has a solution or
  - $P(n - 1, K - k_n)$ does, provided $K - k_n \geq 0$.

The Knapsack Problem (cont.)

An example of the table constructed for the knapsack problem:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	O	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_1 = 2$	O	-	I	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_2 = 3$	O	-	O	I	-	I	-	-	-	-	-	-	-	-	-	-	-
$k_3 = 5$	O	-	O	O	-	O	-	I	I	-	I	-	-	-	-	-	-
$k_4 = 6$	O	-	O	O	-	O	I	O	O	I	O	I	-	I	I	-	I

"I": a solution containing this item has been found.
"O": a solution without this item has been found.
"-": no solution has yet been found.

Source: adapted from [Manber 1989, Figure 5.11].

The Knapsack Problem (cont.)

Algorithm Knapsack (S, K);
    $P[0,0].exist := true$;
    for $k := 1$ **to** K **do**
        $P[0,k].exist := false$;
    for $i := 1$ **to** n **do**
        for $k := 0$ **to** K **do**
            $P[i,k].exist := false$;
            if $P[i-1,k].exist$ **then**
                $P[i,k].exist := true$;
                $P[i,k].belong := false$
            else if $k - S[i] \geq 0$ **then**
                    if $P[i-1,k-S[i]].exist$ **then**
                        $P[i,k].exist := true$;
                        $P[i,k].belong := true$