

# Design by Induction

(Based on [Manber 1989])

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# Introduction

- 🌐 It is **not** necessary to design the steps required to solve a problem **from scratch**.
- 🌐 It is sufficient to guarantee the following:
  1. It is possible to solve one small instance or a few small instances of the problem. (**base case**)
  2. A solution to every problem/instance can be constructed from solutions to smaller problems/instances. (**inductive step**)

# Evaluating Polynomials

## Problem

Given a sequence of real numbers  $a_n, a_{n-1}, \dots, a_1, a_0$ , and a real number  $x$ , compute the value of the polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

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Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

# Evaluating Polynomials (cont.)

Let  $P_{n-1}(x) = a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ .

**Induction hypothesis** (first attempt)

We know how to evaluate a polynomial represented by the input  $a_{n-1}, \cdots, a_1, a_0$ , at the point  $x$ , i.e., we know how to compute  $P_{n-1}(x)$ .

$P_n(x) = a_nx^n + P_{n-1}(x)$ .

# Evaluating Polynomials (cont.)

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$P_n(x) = a_nx^n + P_{n-1}(x)$ .


Number of multiplications:

$$n + (n - 1) + \cdots + 2 + 1 = \frac{n(n + 1)}{2}.$$

# Evaluating Polynomials (cont.)

## **Induction hypothesis** (second attempt)


We know how to compute  $P_{n-1}(x)$ , and we know how to compute  $x^{n-1}$ .

  $P_n(x) = a_n x(x^{n-1}) + P_{n-1}(x).$

# Evaluating Polynomials (cont.)

## **Induction hypothesis** (second attempt)

We know how to compute  $P_{n-1}(x)$ , and we know how to compute  $x^{n-1}$ .

  $P_n(x) = a_n x(x^{n-1}) + P_{n-1}(x).$

 Number of multiplications:  $2n - 1$ .



# Evaluating Polynomials (cont.)


Let  $P'_{n-1}(x) = a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1$ .

**Induction hypothesis** (final attempt)

We know how to evaluate a polynomial represented by the coefficients  $a_n, a_{n-1}, \cdots, a_1$ , at the point  $x$ , i.e., we know how to compute  $P'_{n-1}(x)$ .

$P_n(x) = P'_n(x) = P'_{n-1}(x) \cdot x + a_0$ .

# Evaluating Polynomials (cont.)

 More generally,

$$\begin{cases} P'_0(x) = a_n \\ P'_i(x) = P'_{i-1}(x) \cdot x + a_{n-i}, \text{ for } 1 \leq i \leq n \end{cases}$$

# Evaluating Polynomials (cont.)

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$$\begin{cases} P'_0(x) = a_n \\ P'_i(x) = P'_{i-1}(x) \cdot x + a_{n-i}, \text{ for } 1 \leq i \leq n \end{cases}$$

🌐 Number of multiplications:  $n$ .

# Evaluating Polynomials (cont.)

```
Algorithm Polynomial_Evaluation ( $\bar{a}, x$ );  
begin  
     $P := a_n$ ;  
    for  $i := 1$  to  $n$  do  
         $P := x * P + a_{n-i}$   
end
```

This algorithm is known as *Horner's rule*.

# Maximal Induced Subgraph

## Problem

*Given an undirected graph  $G = (V, E)$  and an integer  $k$ , find an induced subgraph  $H = (U, F)$  of  $G$  of maximum size such that all vertices of  $H$  have degree  $\geq k$  (in  $H$ ), or conclude that no such induced subgraph exists.*

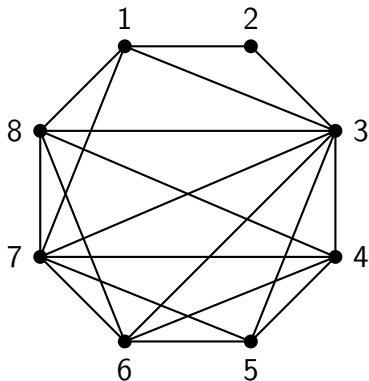
# Maximal Induced Subgraph

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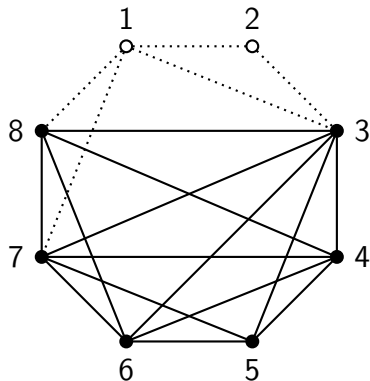
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Design Idea: in the inductive step, we try to **remove one vertex** (that cannot possibly be part of the solution) to get a smaller instance.

# Maximal Induced Subgraph (cont.)



A graph  $G$  of eight nodes.



Maximal induced subgraph of  $G$   
when  $k = 4$ .

# Maximal Induced Subgraph (cont.)

 Recursive:

**Algorithm Max\_Ind\_Subgraph** ( $G, k$ );

**begin**

**if** the degree of every vertex of  $G \geq k$  **then**

        Max\_Ind\_Subgraph :=  $G$ ;

**else** let  $v$  be a vertex of  $G$  with degree  $< k$ ;

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🌐 Iterative:

**Algorithm Max\_Ind\_Subgraph** ( $G, k$ );

**begin**

**while** the degree of some vertex  $v$  of  $G < k$  **do**

$G := G - v$ ;

    Max\_Ind\_Subgraph :=  $G$ ;

**end**

# One-to-One Mapping

## Problem

Given a finite set  $A$  and a mapping  $f$  from  $A$  to itself, find a subset  $S \subseteq A$  with the maximum number of elements, such that

- (1) the function  $f$  maps every element of  $S$  to another element of  $S$  (i.e.,  $f$  maps  $S$  into itself), and
- (2) no two elements of  $S$  are mapped to the same element (i.e.,  $f$  is one-to-one when restricted to  $S$ ).

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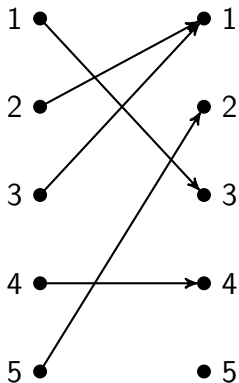
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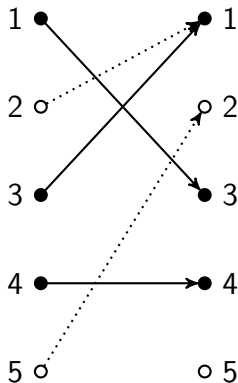
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An element that is not mapped to may be removed.

# One-to-One Mapping (cont.)



A given set  $A$  and  
a mapping to itself.



The maximal selected subset  $S$  and  
the remaining 1-to-1 mapping.

# One-to-One Mapping (cont.)

```
Algorithm Mapping ( $f, n$ );  
begin  
   $S := A$ ;  
  for  $j := 1$  to  $n$  do  $c[j] := 0$ ;  
  for  $j := 1$  to  $n$  do increment  $c[f[j]]$ ;  
  for  $j := 1$  to  $n$  do  
    if  $c[j] = 0$  then put  $j$  in Queue;  
  while Queue not empty do  
    remove  $i$  from the top of Queue;  
     $S := S - \{i\}$ ;  
    decrement  $c[f[i]]$ ;  
    if  $c[f[i]] = 0$  then put  $f[i]$  in Queue  
end
```

## Problem

*Given an  $n \times n$  adjacency matrix, determine whether there exists an  $i$  (the “celebrity”) such that all the entries in the  $i$ -th column (except for the  $ii$ -th entry) are 1, and all the entries in the  $i$ -th row (except for the  $ii$ -th entry) are 0.*

Note: A celebrity corresponds to a **sink** of the directed graph.

Note: Every directed graph has **at most one** sink.

# Celebrity

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Motivation: the trivial solution has a time complexity of  $O(n^2)$ . Can we do better, in  $O(n)$ ?



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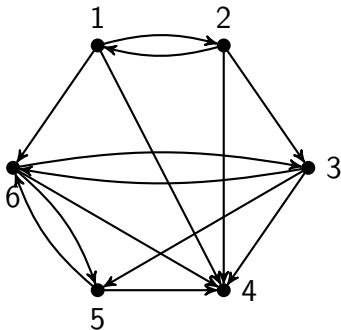
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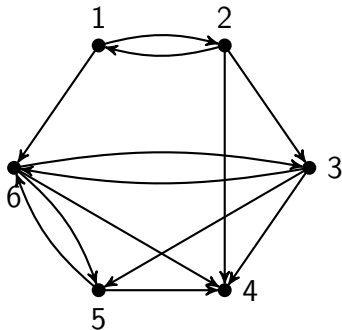
Motivation: the trivial solution has a time complexity of  $O(n^2)$ . Can we do better, in  $O(n)$ ?

To achieve  $O(n)$  time, we must reduce the problem size by at least one in constant time.

# Celebrity (cont.)



A graph of six nodes with a sink (node 4).



A graph of six nodes without a sink.

# Celebrity (cont.)

Basic idea: check whether  $i$  knows  $j$ .

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In either case, one of the two may be eliminated.

The  $O(n)$  algorithm proceeds in two stages:

- 🌐 Eliminate a node every round until only one is left.
- 🌐 Check whether the remaining one is truly a celebrity.

# Celebrity (cont.)

```
Algorithm Celebrity (Know);  
begin  
   $i := 1$ ;  
   $j := 2$ ;  
   $next := 3$ ;  
  while  $next \leq n + 1$  do  
    if  $Know[i, j]$  then  $i := next$   
      else  $j := next$ ;  
     $next := next + 1$ ;  
  if  $i = n + 1$  then  $candidate := j$   
    else  $candidate := i$ ;
```

## Celebrity (cont.)

```
wrong := false;  
k := 1;  
Know[candidate, candidate] := false;  
while not wrong and  $k \leq n$  do  
    if Know[candidate, k] then wrong := true;  
    if not Know[k, candidate] then  
        if candidate  $\neq k$  then wrong := true;  
        k := k + 1;  
if not wrong then celebrity := candidate  
    else celebrity := 0;  
end
```

# The Skyline Problem

## Problem

*Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimension) of these buildings, eliminating hidden lines.*

Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.



# The Skyline Problem

## Problem

*Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimension) of these buildings, eliminating hidden lines.*

Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

Compare: **adding buildings one by one** to an existing skyline **vs.** **merging two skylines** of about the same size

# The Skyline Problem

- 🌐 Adding one building at a time:

$$\begin{cases} T(1) = O(1) \\ T(n) = T(n-1) + O(n), n \geq 2 \end{cases}$$

Time complexity:  $O(n^2)$ .

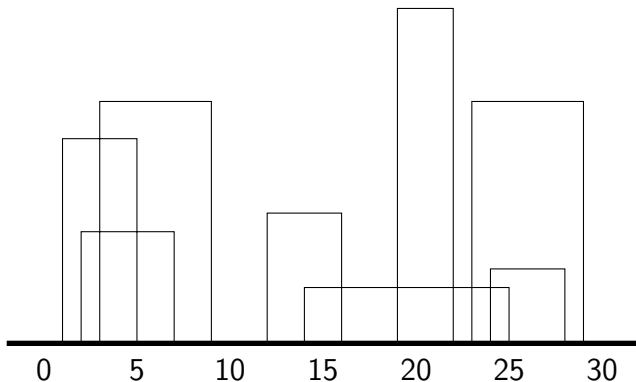
- 🌐 Merging two skylines every round:

$$\begin{cases} T(1) = O(1) \\ T(n) = 2T(\frac{n}{2}) + O(n), n \geq 2 \end{cases}$$

Time complexity:  $O(n \log n)$ .

# Representation of a Skyline

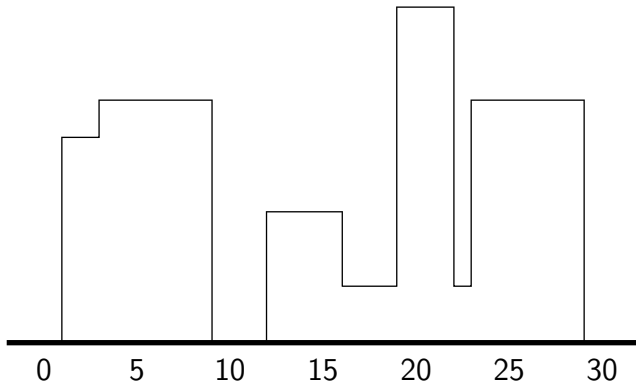
Input:  $(1, \mathbf{11}, 5)$ ,  $(2, \mathbf{6}, 7)$ ,  $(3, \mathbf{13}, 9)$ ,  $(12, \mathbf{7}, 16)$ ,  $(14, \mathbf{3}, 25)$ ,  $(19, \mathbf{18}, 22)$ ,  
 $(23, \mathbf{13}, 29)$ , and  $(24, \mathbf{4}, 28)$ .



Source: adapted from [Manber 1989, Figure 5.5(a)].

## Representation of a Skyline (cont.)

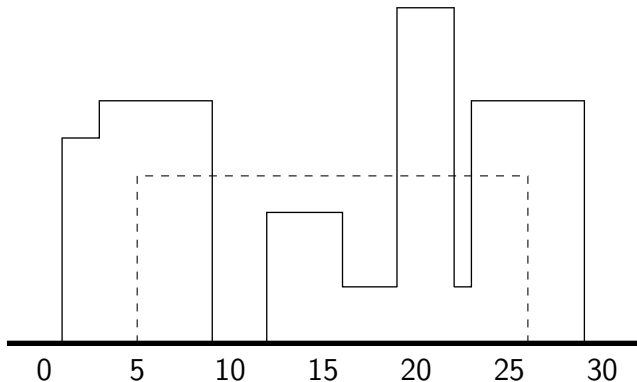
Representation: (1,11,3,13,9,0,12,7,16,3,19,18,22,3,23,13,29).



Source: adapted from [Manber 1989, Figure 5.5(b)].

# Adding a Building

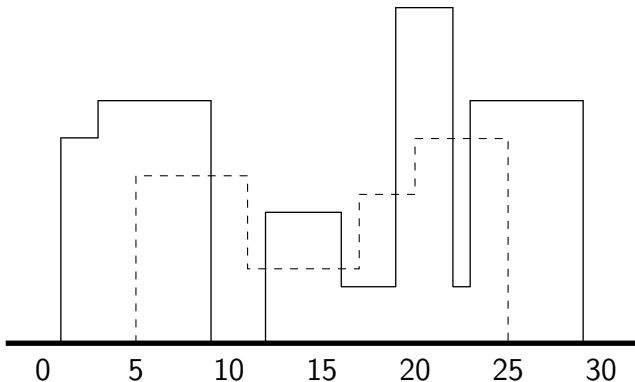
🌐 Add  $(5, 9, 26)$  to  $(1, 11, 3, 13, 9, 0, 12, 7, 16, 3, 19, 18, 22, 3, 23, 13, 29)$ .



Source: adapted from [Manber 1989, Figure 5.6].

🌐 The skyline becomes  $(1, 11, 3, 13, 9, 9, 19, 18, 22, 9, 23, 13, 29)$ .

# Merging Two Skylines



Source: adapted from [Manber 1989, Figure 5.7].

# Balance Factors in Binary Trees

## Problem

*Given a binary tree  $T$  with  $n$  nodes, compute the balance factors of all nodes.*

The **balance factor** of a node is defined as the **difference** between the height of the node's left subtree and the height of the node's right subtree.

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The **balance factor** of a node is defined as the **difference** between the height of the node's left subtree and the height of the node's right subtree.

Motivation: an example of why we must **strengthen the hypothesis** (and hence the problem to be solved).



# Balance Factors in Binary Trees (cont.)

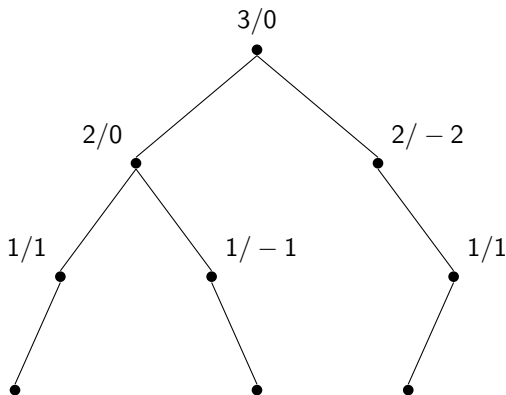


Figure: A binary tree. The numbers represent  $h/b$ , where  $h$  is the height and  $b$  is the balance factor.

Source: redrawn from [Manber 1989, Figure 5.8].

# Balance Factors in Binary Trees (cont.)

## Induction hypothesis

We know how to compute balance factors of all nodes in trees that have  $< n$  nodes.

# Balance Factors in Binary Trees (cont.)

## Induction hypothesis

We know how to compute balance factors of all nodes in trees that have  $< n$  nodes.

## Stronger induction hypothesis

We know how to compute balance factors **and heights** of all nodes in trees that have  $< n$  nodes.

# Maximum Consecutive Subsequence

## Problem

*Given a sequence  $x_1, x_2, \dots, x_n$  of real numbers (not necessarily positive), find a subsequence  $x_i, x_{i+1}, \dots, x_j$  (of consecutive elements) such that the sum of the numbers in it is maximum over all subsequences of consecutive elements.*

Example:

In the sequence  $(2, -3, 1.5, -1, 3, -2, -3, 3)$ , the maximum subsequence is  $(1.5, -1, 3)$ .

# Maximum Consecutive Subsequence

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*Given a sequence  $x_1, x_2, \dots, x_n$  of real numbers (not necessarily positive), find a subsequence  $x_i, x_{i+1}, \dots, x_j$  (of consecutive elements) such that the sum of the numbers in it is maximum over all subsequences of consecutive elements.*

Example:

In the sequence  $(2, -3, 1.5, -1, 3, -2, -3, 3)$ , the maximum subsequence is  $(1.5, -1, 3)$ .

Motivation: another example of strengthening the hypothesis.

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We know how to find the maximum subsequence in sequences of size  $< n$ .

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We know how to find, in sequences of size  $< n$ , the maximum subsequence overall and the maximum subsequence that is a suffix.



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We know how to find the maximum subsequence in sequences of size  $< n$ .

## Stronger induction hypothesis

We know how to find, in sequences of size  $< n$ , the maximum subsequence overall **and the maximum subsequence that is a suffix**.

Reasoning: the maximum subsequence of problem size  $n$  is obtained either

-  directly from the maximum subsequence of problem size  $n - 1$  or
-  from appending the  $n$ -th element to the maximum suffix of problem size  $n - 1$ .



# Maximum Consecutive Subsequence (cont.)

```
Algorithm Max_Consec_Subseq ( $X, n$ );  
begin  
   $Global\_Max := 0$ ;  
   $Suffix\_Max := 0$ ;  
  for  $i := 1$  to  $n$  do  
    if  $x[i] + Suffix\_Max > Global\_Max$  then  
       $Suffix\_Max := Suffix\_Max + x[i]$ ;  
       $Global\_Max := Suffix\_Max$   
    else if  $x[i] + Suffix\_Max > 0$  then  
       $Suffix\_Max := Suffix\_Max + x[i]$   
    else  $Suffix\_Max := 0$   
end
```

# The Knapsack Problem

## Problem

*Given an integer  $K$  and  $n$  items of different sizes such that the  $i$ -th item has an integer size  $k_i$ , find a subset of the items whose sizes sum to exactly  $K$ , or determine that no such subset exists.*

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Design Idea: use **strong induction** so that solutions to **all smaller instances** may be used.

# The Knapsack Problem (cont.)

- Let  $P(n, K)$  denote the problem where  $n$  is the number of items and  $K$  is the size of the knapsack.
- Induction hypothesis**  
We know how to solve  $P(n - 1, K)$ .

# The Knapsack Problem (cont.)

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- Induction hypothesis**  
We know how to solve  $P(n - 1, K)$ .
- Stronger induction hypothesis**  
We know how to solve  $P(n - 1, k)$ , for all  $0 \leq k \leq K$ .

# The Knapsack Problem (cont.)

🌐 Let  $P(n, K)$  denote the problem where  $n$  is the number of items and  $K$  is the size of the knapsack.

🌐 **Induction hypothesis**

We know how to solve  $P(n - 1, K)$ .

🌐 **Stronger induction hypothesis**

We know how to solve  $P(n - 1, k)$ , for all  $0 \leq k \leq K$ .

Reasoning:  $P(n, K)$  has a solution if either

- ☀️  $P(n - 1, K)$  has a solution or
- ☀️  $P(n - 1, K - k_n)$  does, provided  $K - k_n \geq 0$ .

# The Knapsack Problem (cont.)

An example of the table constructed for the knapsack problem:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	O	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_1 = 2$	O	-	I	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_2 = 3$	O	-	O	I	-	I	-	-	-	-	-	-	-	-	-	-	-
$k_3 = 5$	O	-	O	O	-	O	-	I	I	-	I	-	-	-	-	-	-
$k_4 = 6$	O	-	O	O	-	O	I	O	O	I	O	I	-	I	I	-	I

“I”: a solution containing this item has been found.

“O”: a solution without this item has been found.

“-”: no solution has yet been found.

Source: adapted from [Manber 1989, Figure 5.11].

# The Knapsack Problem (cont.)

**Algorithm Knapsack** ( $S, K$ );

$P[0, 0].exist := true$ ;

**for**  $k := 1$  **to**  $K$  **do**

$P[0, k].exist := false$ ;

**for**  $i := 1$  **to**  $n$  **do**

**for**  $k := 0$  **to**  $K$  **do**

$P[i, k].exist := false$ ;

**if**  $P[i - 1, k].exist$  **then**

$P[i, k].exist := true$ ;

$P[i, k].belong := false$

**else if**  $k - S[i] \geq 0$  **then**

**if**  $P[i - 1, k - S[i]].exist$  **then**

$P[i, k].exist := true$ ;

$P[i, k].belong := true$