Algorithms 2023: Searching and Sorting

(Based on [Manber 1989])

Yih-Kuen Tsay

October 18, 2023

1 Binary Search

Searching a Sorted Sequence

Problem 1. *Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.*

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

function Find $(z, Left, Right) : integer$;
begin
  if $Left = Right$ **then**
    if $X[Left] = z$ **then** $Find := Left$
    else $Find := 0$
  else
    $Middle := \lceil \frac{Left+Right}{2} \rceil$;
    if $z < X[Middle]$ **then**
      $Find := Find(z, Left, Middle - 1)$
    else
      $Find := Find(z, Middle, Right)$
end

Algorithm Binary_Search (X, n, z);
begin
  $Position := Find(z, 1, n)$;
end

Binary Search: Alternative

```
function Find (z, Left, Right) : integer;
begin
    if Left > Right then
        Find := 0
    else
        Middle := ⌈(Left+Right)/2⌉;
        if z = X[Middle] then
            Find := Middle
        else if z < X[Middle] then
            Find := Find(z, Left, Middle − 1)
        else
            Find := Find(z, Middle + 1, Right)
end
```

How do the two algorithms compare?

/* The alternative may stop early once the target is found at *Middle*; otherwise, it spends another comparison to divide the search space. If by experience you expect to find the target almost all of the time, then consider using the alternative algorithm. */

1.1 Cyclically Sorted Sequence

Searching a Cyclically Sorted Sequence

Problem 2. *Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).*

- Example 1:
	1	2	3	4	5	6	7	8	
[	5	6	7	0	1	2	3	4	]
  - The 4th is the minimal element.
- Example 2:
	1	2	3	4	5	6	7	8	
[	0	1	2	3	4	5	6	7	]
  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

  /* If $X[Middle] < X[Right]$, then the minimal is in the left half (including $X[Middle]$); otherwise, it is in the right half (excluding $X[Middle]$). */

Cyclic Binary Search

```
Algorithm Cyclic_Binary_Search (X, n);
begin
    Position := Cyclic_Find(1, n);
end

function Cyclic_Find (Left, Right) : integer;
begin
    if Left = Right then Cyclic_Find := Left
```

```
    else
        Middle := ⌊(Left+Right)/2⌋;
        if X[Middle] < X[Right] then
            Cyclic_Find := Cyclic_Find(Left, Middle)
        else
            Cyclic_Find := Cyclic_Find(Middle + 1, Right)
end
```

1.2 "Fixpoints"

"Fixpoints"

Problem 3. *Given a sorted sequence of* distinct *integers* $a_1, a_2, \cdots, a_n$*, determine whether there exists an index* i *such that* $a_i = i$*.*

- Example 1:
  - $\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$
  - $a_4 = 4$ (there are more ...).
- Example 2:
  - $\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$
  - There is no i such that $a_i = i$.
- Again, can we cut the search space in half by asking only one question?

  /* As the numbers are distinct, they increase or decrease at least as fast as the indices (which always increase or decrease by one). If $X[Middle] < Middle$, then the fixpoint (if it exists) must be in the right half (excluding $X[Middle]$); otherwise, it must be in the left half (including $X[Middle]$). */

A Special Binary Search

```
function Special_Find (Left, Right) : integer;
begin
    if Left = Right then
        if A[Left] = Left then Special_Find := Left
        else Special_Find := 0
    else
        Middle := ⌊(Left+Right)/2⌋;
        if A[Middle] < Middle then
            Special_Find := Special_Find(Middle + 1, Right)
        else
            Special_Find := Special_Find(Left, Middle)
end
```

A Special Binary Search (cont.)

```
Algorithm Special_Binary_Search (A, n);
begin
    Position := Special_Find(1, n);
end
```

1.3 Stuttering Subsequence

Stuttering Subsequence

Problem 4. *Given two sequences A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the maximal value of i such that B^i is a subsequence of A.*

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.

  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.

  Time complexity: $O(n \log \frac{n}{m})$.

Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

/* Try 2^0, 2^1, 2^2, $\cdots$, 2^{k-1}, and 2^k sequentially. If the target falls between 2^{k-1} and 2^k, apply binary search within that region. */

2 Interpolation Search

Interpolation Search

/* Interpolation search is a refined version of binary search. The basic idea is that, if the target value is closer to the value of the left (right) element, then one should divide the array at a point closer to the left (right) boundary. */

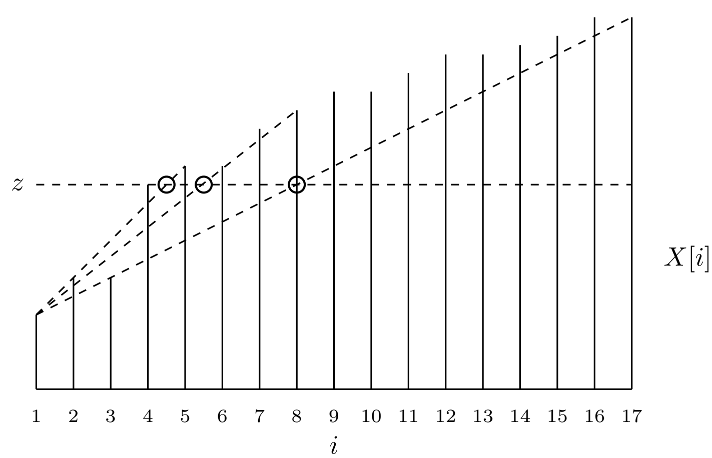
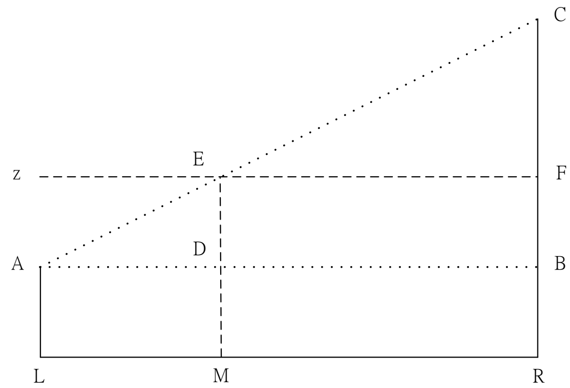


Figure: Interpolation search.

Source: redrawn from [Manber 1989, Figure 6.4].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

function Int_Find $(z, Left, Right) : integer$;
begin
  if $X[Left] = z$ **then** $Int\_Find := Left$
  else if $Left = Right$ or $X[Left] = X[Right]$ **then**
    $Int\_Find := 0$
  else
    $Next\_Guess := \lceil Left + \frac{(z-X[Left])(Right-Left)}{X[Right]-X[Left]} \rceil$;
    if $z < X[Next\_Guess]$ **then**
      $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
    else
      $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

/* $Next\_Guess - Left = |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}| \approx \lceil \frac{(z-X[Left])(Right-Left)}{X[Right]-X[Left]} \rceil$ */

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
  if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
  else $Position := Int\_Find(z, 1, n)$;
end

3 Sorting

Sorting

Problem 5. *Given n numbers x_1, x_2, $\cdots$, x_n, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \le i_1, i_2, \cdots, i_n \le n$, such that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$.*

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

3.1 Using Balanced Search Trees

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
  1. Create an empty tree.
  2. Insert the numbers one by one to the tree.
  3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

  /* The time complexity is $O(n \log n)$, as there are n elements to insert and each insertion takes $O(\log n)$ time. Traversal of the tree can be done efficiently in $O(n)$ time. */

3.2 Radix Sort

Radix Sort

/* Radix Sort algorithms assume that a number is composed of digits, each of which may be examined separately, unlike most other sorting algorithms where numbers are only compared to determine their relative order. */

```
Algorithm Straight_Radix (X, n, k);
begin
    put all elements of X in a queue GQ;
    for i := 1 to d do /* A digit may be 1, 2, ..., or d. */
        initialize queue Q[i] to be empty
    for i := k downto 1 do
        while GQ is not empty do
                pop x from GQ;
                c := the i-th digit of x;
                insert x into Q[c];
        for t := 1 to d do
            insert Q[t] into GQ;
    for i := 1 to n do
        pop X[i] from GQ
end
```

Time complexity: $O(nk)$.

/* The inductive thinking of Straight Radix Sort goes as follows: to sort the numbers according to the j-th through the k-th digits, supposing we have sorted the numbers according to the $(j+1)$-th through the k-th digits, we just need to sort the numbers according to the j-th digit while preserving the original order for the numbers with the same j-th digit.

Unfold the induction/recursion and present the algorithm in an iterative form, we get the pseudocode as above where the numbers are sorted one digit at a time, from the k-th digit back to the 1-st digit (which is the most significant digit). */

3.3 Merge Sort

Merge Sort

Algorithm Mergesort (X, n);
begin $M\_Sort(1, n)$ **end**

procedure M_Sort $(Left, Right)$;
begin
  if $Right - Left = 1$ **then**
    if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$
  else if $Left \neq Right$ **then**
      $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
      $M\_Sort(Left, Middle - 1)$;
      $M\_Sort(Middle, Right)$;

Merge Sort (cont.)

      $i := Left$; $j := Middle$; $k := 0$;
      while $(i \leq Middle - 1)$ and $(j \leq Right)$ **do**
        $k := k + 1$;
        if $X[i] \leq X[j]$ **then**
          $TEMP[k] := X[i]$; $i := i + 1$
        else $TEMP[k] := X[j]$; $j := j + 1$;
      if $j > Right$ **then**
        for $t := 0$ **to** $Middle - 1 - i$ **do**
          $X[Right - t] := X[Middle - 1 - t]$
      for $t := 0$ **to** $k - 1$ **do**
        $X[Left + t] := TEMP[1 + t]$
end

/* In the merging stage, the while loop terminates when one of the two halves is exhausted. If the left half is exhausted ($j \leq Right$), the remaining elements in the right half are already in the correct positions in Array X and so nothing needs to be done. */

Time complexity: $O(n \log n)$.

/*

$$\begin{cases} T(1) = O(1) \\ T(n) = 2T(\frac{n}{2}) + O(n), n \geq 2 \end{cases}$$

Apply the master theorem with $a = 2$, $b = 2$, $k = 1$, and $b^k = 2 = a$). */

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
②	⑥	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	⑤	⑧	10	9	12	1	15	7	3	13	4	11	16	14
②	⑤	⑥	⑧	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	⑨	⑩	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	①	⑫	15	7	3	13	4	11	16	14
2	5	6	8	①	⑨	⑩	⑫	15	7	3	13	4	11	16	14
①	②	⑤	⑥	⑧	⑨	⑩	⑫	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	⑦	⑮	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	③	⑬	4	11	16	14
1	2	5	6	8	9	10	12	③	⑦	⑬	⑮	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	⑭	⑯
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	⑭	⑯
1	2	5	6	8	9	10	12	③	④	⑦	⑪	⑬	⑭	⑮	⑯
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯

Figure: An example of mergesort.

Source: redrawn from [Manber 1989, Figure 6.8].

/* The table shows the order in which all the merges occur. However, it does not show the movements of the elements from the array to the temporary array and back. */

3.4 Quick Sort

Quick Sort

Algorithm Quicksort (X, n);
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort (*Left*, *Right*);
begin
    if *Left* < *Right* **then**
        $Partition(X, Left, Right)$;
        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

/* The worst-case time complexity $O(n^2)$ occurs when the input array is already sorted or nearly sorted, as each partition (which is of time $O(n)$) will successively divide the array into two parts of sizes 1 and $n-2$, 1 and $n-4$, 1 and $n-6$, etc. This may be avoided by choosing the pivot more "wisely". */

Quick Sort (cont.)

Algorithm Partition$(X, Left, Right)$;
begin
    $pivot := X[Left]$;

```
    L := Left;  R := Right;
    while L < R do
        while X[L] ≤ pivot and L ≤ Right do L := L + 1;
        while X[R] > pivot and R ≥ Left do R := R − 1;
        if L < R then swap(X[L], X[R]);
    Middle := R;
    swap(X[Left], X[Middle])
end
```

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	④	5	10	9	12	1	15	7	3	13	⑧	11	16	14
6	2	4	5	③	9	12	1	15	7	⑩	13	8	11	16	14
6	2	4	5	3	①	12	⑨	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14

Figure: Partition of an array around the pivot 6.

Source: redrawn from [Manber 1989, Figure 6.10].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure: An example of quicksort.

Source: redrawn from [Manber 1989, Figure 6.12].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

  The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\textstyle\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

3.5 Heap Sort

Heap Sort

Algorithm Heapsort (A, n);
begin
  $Build\_Heap(A)$;
  for $i := n$ **downto** 2 **do**
    $swap(A[1], A[i])$;
    $Rearrange\_Heap(i-1)$
end

Time complexity: $O(n \log n)$

/* The complexity is so, mainly thanks to the efficiency of *Rearrange_Heap*, which is $O(\log n)$. */

Heap Sort (cont.)

procedure Rearrange_Heap (k);
begin
  $parent := 1$;
  $child := 2$;
  while $child \leq k-1$ **do**
    if $A[child] < A[child+1]$ **then**
      $child := child + 1$;
    if $A[child] > A[parent]$ **then**
      $swap(A[parent], A[child])$;
      $parent := child$;
      $child := 2 * child$
    else $child := k$
end

Heap Sort (cont.)

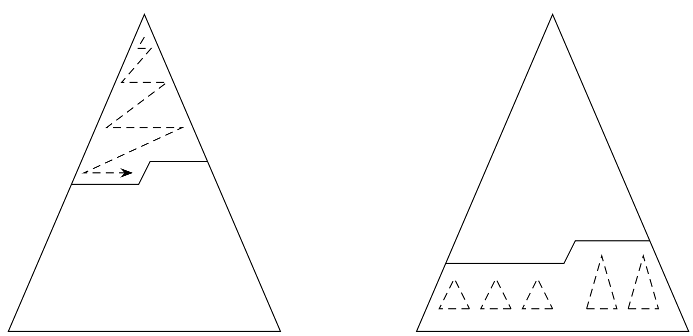


Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

How do the two approaches compare?

/* Top down: $O(n \log n)$.

Bottom up: O(sum of the heights of all nodes) $= O(n)$. Consider a full binary tree of height h. From an excercise problem in HW#2, we know that "sum of the heights of all nodes" of the tree equals $2^{h+1} - (h+2) \leq 2^{h+1} - 1 = n$. */

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	8	5	10	9	12	14	15	7	3	13	4	11	16	1
6	2	8	5	10	9	16	14	15	7	3	13	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	15	10	13	16	14	5	7	3	9	4	11	12	1
6	2	16	15	10	13	12	14	5	7	3	9	4	11	8	1
6	15	16	14	10	13	12	2	5	7	3	9	4	11	8	1
16	15	13	14	10	9	12	2	5	7	3	6	4	11	8	1

Figure: An example of building a heap bottom up.

Source: adapted from [Manber 1989, Figure 6.15].

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem 6 (Theorem 6.1)**.** *Every decision-tree algorithm for sorting has height* $\Omega(n \log n)$.

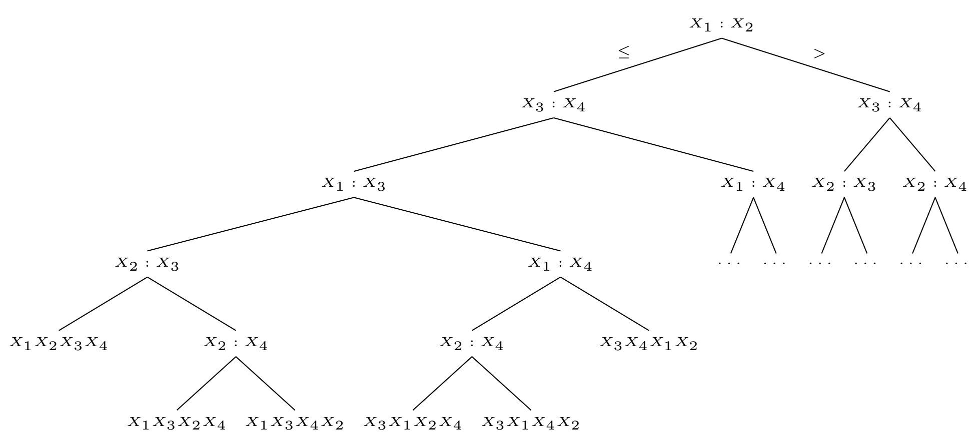
Proof idea: there must be at least $n!$ leaves in the decision tree, one for each possible outcome.

/* Recall Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$. The height of the decision tree must be at least $\log(n!)$, i.e., $\Omega(n \log n)$. */

Is the lower bound contradictory to the time complexity of radix sort?

A Lower Bound for Sorting (cont.)

A decision tree (partly shown) for the merge sort with $X_1X_2X_3X_4$ as input:



Note: in total, there should be $4! = 24$ leaves, only six of which are shown.

4 Order Statistics

Order Statistics: Minimum and Maximum

Problem 7. *Find the maximum and minimum elements in a given sequence.*

- The obvious solution requires $(n-1)+(n-2)$ $(=2n-3)$ comparisons between elements.
- Can we do better? (Which comparisons could have been avoided?)

  /* A better algorithm: compare x_1 and x_2. Set *min* to be the smaller of the two and *max* the larger. Compare x_3 and x_4 and then compare the smaller with *min* and the larger with *max*; these take three comparisons. Update *min* and *max* accordingly. Continue until we have exhausted the sequence of numbers. Assuming n is even, the total number of comparisons $= 1 + 3 \times \frac{(n-2)}{2} = \frac{3}{2}n - 2$.

  Suppose $x_1 < x_2 < x_3 < x_4$. Using the obvious solution to find the minimum and then the maximum, we would make the following five comparisons: $x_1 : x_2$, $x_1 : x_3$, and $x_1 : x_4$ and then $x_2 : x_3$ and $x_3 : x_4$. With the above algorithm, we will make just four comparisons: $x_1 : x_2$, and then $x_3 : x_4$, $x_1 : x_3$, and $x_2 : x_4$. In particular, the comparison $x_1 : x_4$ (whose result may be inferred from $x_1 : x_3$ and $x_3 : x_4$) in the obvious solution has been avoided. */

Order Statistics: Kth-Smallest

Problem 8. *Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.*

Order Statistics: Kth-Smallest (cont.)

```
procedure Select (Left, Right, k);
begin
   if Left = Right then
      Select := Left
   else Partition(X, Left, Right);
        let Middle be the output of Partition;
        if Middle − Left + 1 ≥ k then
           Select(Left, Middle, k)
        else
           Select(Middle + 1, Right, k − (Middle − Left + 1))
end
```

```
Algorithm Selection (X, n, k);
begin
   if (k < 1) or (k > n) then print "error"
   else S := Select(1, n, k)
end
```

/* Here the formal parameter k (for rank) is made to be relative to the left bound of array indices, while *Left*, *Middle*, and *Right* are absolute index values. */

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

```
procedure Select (Left, Right, k);
begin
    if Left = Right then
       Select := Left
    else Partition(X, Left, Right);
          let Middle be the output of Partition;
          if Middle ≥ k then
             Select(Left, Middle, k)
          else
             Select(Middle + 1, Right, k)
end
```

5 Finding a Majority

Finding a Majority

Problem 9. *Given a sequence of numbers, find the majority in the sequence or determine that none exists.*

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Caution: maintaining a counter for each possible number requires $O(\log n)$ time for each access to a particular counter, which means $O(n \log n)$ time in total. Sorting the sequence to find a probable candidate also requires $O(n \log n)$ time.

Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

/* If there is a majority, it is also a majority of the other $n-2$ numbers. However, the reverse may not be true. */

What if they are equal?

/* Keep the first number as a candidate at hand and repeat the following:

If the next number equals the candidate, we increment the count of its occurrences; otherwise, we have a pair of unequal numbers to eliminate (by decrementing the count for the candidate). When the count becomes 0 (due to elimination), we take the next number as a new candidate. */

Finding a Majority (cont.)

```
Algorithm Majority (X, n);
begin
    C := X[1];  M := 1;
    for i := 2 to n do
        if M = 0 then
            C := X[i];  M := 1
        else
            if C = X[i] then M := M + 1
            else M := M − 1;
```

Finding a Majority (cont.)

```
if M = 0 then Majority := −1
else
    Count := 0;
    for i := 1 to n do
        if X[i] = C then Count := Count + 1;
    if Count > n/2 then Majority := C
    else Majority := −1
end
```

Time complexity: $O(n)$.