


# Dynamic Programming

(Based on [Cormen et al. 2009])

Yih-Kuen Tsay


Department of Information Management  
National Taiwan University

## Greedy

-  Huffman's encoding algorithm, Dijkstra's algorithm, Prim's algorithm, etc.

# Design Methods

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## Divide-and-Conquer

-  Binary search, merge sort, quick sort, etc.

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## **Dynamic Programming**

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## **Dynamic Programming**

## Branch-and-Bound

## ...

- 🌐 Property of Optimal Substructure (Principle of Optimality):  
*An optimal solution to a problem contains **optimal solutions to its subproblems**.*
- 🌐 A particular subproblem or subsubproblem typically recurs while one tries different decompositions of the original problem.
- 🌐 To reduce running time, optimal solutions to subproblems are **computed only once and stored** (in an array) for subsequent uses.


# Development by Dynamic Programming

1. Characterize the structure of an optimal solution.
2. Recursively define the value of an optimal solution.
3. Compute the value of an optimal solution in a bottom-up fashion.
4. Construct an optimal solution from computed information.

# Matrix-Chain Multiplication

## Problem

*Given a chain  $A_1, A_2, \dots, A_n$  of matrices where  $A_i, 1 \leq i \leq n$ , has dimension  $p_{i-1} \times p_i$ , fully parenthesize (i.e., find a way to evaluate) the product  $A_1 A_2 \dots A_n$  such that the number of scalar multiplications is minimum.*

 Why is dynamic programming a feasible approach?



# Matrix-Chain Multiplication

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Given a chain  $A_1, A_2, \dots, A_n$  of matrices where  $A_i$ ,  $1 \leq i \leq n$ , has dimension  $p_{i-1} \times p_i$ , fully parenthesize (i.e., find a way to evaluate) the product  $A_1 A_2 \dots A_n$  such that the number of scalar multiplications is minimum.

- 🌐 Why is dynamic programming a feasible approach?
- 🌐 To evaluate  $A_1 A_2 \dots A_n$ , one first has to evaluate  $A_1 A_2 \dots A_k$  and  $A_{k+1} A_{k+2} \dots A_n$  for some  $k$  and then multiply the two resulting matrices.
- 🌐 An optimal way for evaluating  $A_1 A_2 \dots A_n$  must contain optimal ways for evaluating  $A_1 A_2 \dots A_k$  and  $A_{k+1} A_{k+2} \dots A_n$  for some  $k$ .

# Matrix-Chain Multiplication (cont.)

Let  $m[i, j]$  be the minimum number of scalar multiplications needed to compute  $A_i A_{i+1} \cdots A_j$ , where  $1 \leq i \leq j \leq n$ .

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

# Matrix-Chain Multiplication (cont.)

```
Algorithm Matrix_Chain_Order( $n, p$ );  
begin  
  for  $i := 1$  to  $n$  do  
     $m[i, i] := 0$ ;  
  for  $l := 2$  to  $n$  do {  $l$  is the chain length }  
    for  $i := 1$  to  $(n - l + 1)$  do  
       $j := i + l - 1$ ;  
       $m[i, j] := \infty$ ;  
      for  $k := i$  to  $(j - 1)$  do  
        if  $m[i, k] + m[k + 1, j] + p[i - 1]p[k]p[j] < m[i, j]$  then  
           $m[i, j] := m[i, k] + m[k + 1, j] + p[i - 1]p[k]p[j]$   
end
```

# Recursive Implementation

```
Algorithm Recursive_Matrix_Chain( $p, i, j$ );  
begin  
  if  $i = j$  then return 0;  
   $m[i, j] := \infty$ ;  
  for  $k := i$  to  $(j - 1)$  do  
     $q :=$  Recursive_Matrix_Chain( $p, i, k$ ) +  
      Recursive_Matrix_Chain( $p, k + 1, j$ ) +  $p[i - 1]p[k]p[j]$ ;  
    if  $q < m[i, j]$  then  
       $m[i, j] := q$ ;  
  return  $m[i, j]$   
end
```

# Recursive Implementation (cont.)

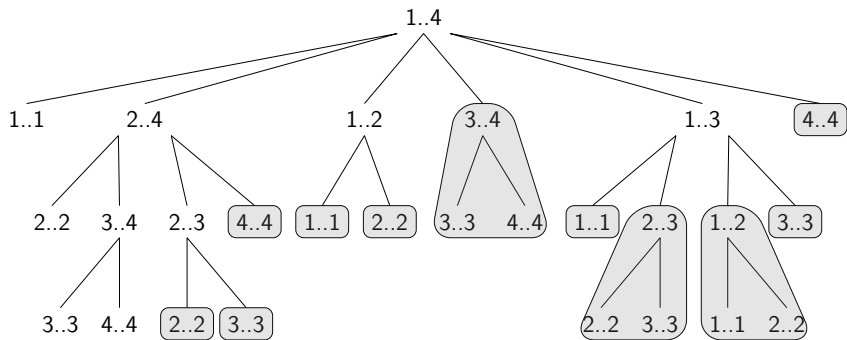


Figure: The recursion tree for the computation of **Recursive\_Matrix\_Chain**( $p, 1, 4$ ). The computations performed in a shaded subtree are replaced by a table lookup.

Source: redrawn from [Cormen *et al.* 2006, Figure 15.5].

# Recursion with Memoization

```
Algorithm Memoized_Matrix_Chain( $n, p$ );  
begin  
  for  $i := 1$  to  $n$  do  
    for  $j := i$  to  $n$  do  
       $m[i, j] := \infty$ ;  
    return Lookup_Matrix_Chain( $p, i, n$ )  
end
```

## Recursion with Memoization (cont.)

```
Function Lookup_Matrix_Chain( $p, i, j$ );  
begin  
  if  $m[i, j] < \infty$  then return  $m[i, j]$ ;  
  if  $i = j$  then  
     $m[i, j] := 0$ ;  
  else  
    for  $k := i$  to  $(j - 1)$  do  
       $q :=$  Lookup_Matrix_Chain( $p, i, k$ ) +  
        Lookup_Matrix_Chain( $p, k + 1, j$ ) +  $p[i - 1]p[k]p[j]$ ;  
      if  $q < m[i, j]$  then  
         $m[i, j] := q$ ;  
  return  $m[i, j]$   
end
```

# Single-Source Shortest Paths

## Problem

Given a weighted directed graph  $G = (V, E)$  with no negative-weight cycles and a vertex  $v$ , find (the lengths of) the shortest paths from  $v$  to all other vertices.

- 🌐 Notice that edges with negative weights are permitted; so, Dijkstra's algorithm does not work here.
- 🌐 A shortest path from  $v$  to any other vertex  $u$  contains at most  $n - 1$  edges.
- 🌐 A shortest path from  $v$  to  $u$  with at most  $k (> 1)$  edges is either (1) a known shortest path from  $v$  to  $u$  with at most  $k - 1$  edges or (2) composed of a shortest path from  $v$  to  $u'$  with at most  $k - 1$  edges and the edge from  $u'$  to  $u$ , for some  $u'$ .



# Single-Source Shortest Paths (cont.)

Denote by  $D^l(u)$  the length of a shortest path from  $v$  to  $u$  containing *at most*  $l$  edges; particularly,  $D^{n-1}(u)$  is the length of a shortest path from  $v$  to  $u$  (with no restrictions).

$$D^1(u) = \begin{cases} \text{length}(v, u) & \text{if } (v, u) \in E \\ 0 & \text{if } u = v \\ \infty & \text{otherwise} \end{cases}$$

$$D^l(u) = \min\{D^{l-1}(u), \min_{(u', u) \in E} \{D^{l-1}(u') + \text{length}(u', u)\}\}, \\ 2 \leq l \leq n - 1$$

# Single-Source Shortest Paths (cont.)

**Algorithm Single\_Source\_Shortest\_Paths\_withL(*length*);**

**begin**

$D[1][v] := 0;$

**for all  $u \neq v$  do**

**if  $(v, u) \in E$  then**

$D[1][u] := \text{length}(v, u)$

**else  $D[1][u] := \infty;$**

**for  $l := 2$  to  $n - 1$  do**

**for all  $u$  do**

$D[l][u] := D[l - 1][u];$

**for all  $u \neq v$  do**

**for all  $u'$  such  $(u', u) \in E$  do**

**if  $D[l - 1][u'] + \text{length}[u', u] < D[l][u]$  then**

$D[l][u] := D[l - 1][u'] + \text{length}[u', u]$

**end**

# Single-Source Shortest Paths (cont.)

```
Algorithm Single_Source_Shortest_Paths(length);  
begin  
   $D[v] := 0$ ;  
  for all  $u \neq v$  do  
    if  $(v, u) \in E$  then  
       $D[u] := \text{length}(v, u)$   
    else  $D[u] := \infty$ ;  
  for  $l := 2$  to  $n - 1$  do  
    for all  $u \neq v$  do  
      for all  $u'$  such  $(u', u) \in E$  do  
        if  $D[u'] + \text{length}[u', u] < D[u]$  then  
           $D[u] := D[u'] + \text{length}[u', u]$   
end
```

# All-Pairs Shortest Paths

## Problem

*Given a weighted directed graph  $G = (V, E)$  with no negative-weight cycles, find (the lengths of) the shortest paths between all pairs of vertices.*

- 🌐 Consider a shortest path from  $v_i$  to  $v_j$  and an arbitrary intermediate vertex  $v_k$  (if any) on this path.
- 🌐 The subpath from  $v_i$  to  $v_k$  must also be a shortest path from  $v_i$  to  $v_k$ ; analogously for the subpath from  $v_k$  to  $v_j$ .

## All-Pairs Shortest Paths (cont.)

Index the vertices from 1 through  $n$ .

Denote by  $W^k(i, j)$  the length of a shortest path from  $v_i$  to  $v_j$  going through no vertex of index greater than  $k$ , where  $1 \leq i, j \leq n$  and  $0 \leq k \leq n$ ; particularly,  $W^n(i, j)$  is the length of a shortest path from  $v_i$  to  $v_j$ .

$$W^0(i, j) = \begin{cases} \text{length}(i, j) & \text{if } (i, j) \in E \\ 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

$$W^k(i, j) = \min\{W^{k-1}(i, j), W^{k-1}(i, k) + W^{k-1}(k, j)\}, 1 \leq k \leq n$$

Note:  $W^k(i, j)$  is the length of the shortest ( $\leq k$ )-path from  $v_i$  to  $v_j$  (see the slides for basic graph algorithms).

# All-Pairs Shortest Paths (cont.)

**Algorithm All\_Pairs\_Shortest\_Paths\_withK**(*length*);

**begin**

**for**  $i := 1$  to  $n$  **do**

**for**  $j := 1$  to  $n$  **do**

**if**  $(i, j) \in E$  **then**  $W[0][i, j] := \text{length}(i, j)$

**else**  $W[0][i, j] := \infty$ ;

**for**  $i := 1$  to  $n$  **do**  $W[0][i, i] := 0$ ;

**for**  $k := 1$  to  $n$  **do**

**for**  $i := 1$  to  $n$  **do**

**for**  $j := 1$  to  $n$  **do**

$W[k][i, j] := W[k - 1][i, j]$ ;

**for**  $i := 1$  to  $n$  **do**

**for**  $j := 1$  to  $n$  **do**

**if**  $W[k - 1][i, k] + W[k - 1][k, j] < W[k][i, j]$  **then**

$W[k][i, j] := W[k - 1][i, k] + W[k - 1][k, j]$

**end**

# All-Pairs Shortest Paths (cont.)

```
Algorithm All_Pairs_Shortest_Paths(length);  
begin  
  for  $i := 1$  to  $n$  do  
    for  $j := 1$  to  $n$  do  
      if  $(i, j) \in E$  then  $W[i, j] := \text{length}(i, j)$   
      else  $W[i, j] := \infty$ ;  
  for  $i := 1$  to  $n$  do  $W[i, i] := 0$ ;  
  for  $k := 1$  to  $n$  do  
    for  $i := 1$  to  $n$  do  
      for  $j := 1$  to  $n$  do  
        if  $W[i, k] + W[k, j] < W[i, j]$  then  
           $W[i, j] := W[i, k] + W[k, j]$   
end
```