Homework Assignment #2

Due Time/Date

2:20PM Tuesday, September 19, 2023. Late submission will be penalized by 20% for each working day overdue.

How to Submit

Please write or type your answers on A4 (or similar size) paper. Drop your homework by the due time in Yih-Kuen Tsay's mail box on the first floor of Management College Building 2. You may discuss the problems with others, but copying answers is strictly forbidden.

Problems

There are five problems in this assignment, each accounting for 20 points. You must use *induction* for all proofs. (Note: problems marked with "(X.XX)" are taken from [Manber 1989] with probable adaptation.)

1. Consider again the inductive definition in HW#1 for the set of all binary trees that store non-negative integer key values:

   - The empty tree, denoted $\perp$, is a binary tree, storing no key value.
   - If t_l and t_r are binary trees, then $node(k, t_l, t_r)$, where $k \in \mathbb{Z}$ and $k \geq 0$, is a also binary tree with the root storing key value k.

   (a) Refine the definition to include only binary *search* trees where an inorder traversal of a binary search tree produces a list of all stored key values in *increasing* order.

   (b) Further refine the definition to include only AVL trees, which are binary search trees where the heights of the left and the right children of every internal node differ by at most 1.

2. (2.30) A **full binary tree** is defined inductively as follows. (Note: some authors prefer using the name "perfect binary tree" or "complete binary tree", while reserving "full binary tree" for another variant of binary trees.) A full binary tree of height 0 consists of 1 node which is the root. A full binary tree of height $h + 1$ consists of two full binary trees of height h whose roots are connected to a new root. Let T be a full binary tree of height h. The **height** of a node in T is h minus the node's distance from the root (e.g., the root has height h, whereas a leaf has height 0). Prove that the sum of the heights of all the nodes in T is $2^{h+1} - h - 2$.

3. (2.23) The **lattice** points in the plane are the points with integer coordinates. Let P be a polygon that does not cross itself (such a polygon is called **simple**) such that all of its vertices are lattice points (see Figure 1). Let p be the number of lattice points that are on the boundary of the polygon (including its vertices), and let q be the number of lattice points that are inside the polygon. Prove that the area of polygon is $\frac{p}{2} + q - 1$.

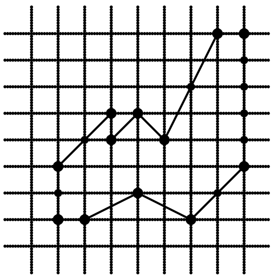

Figure 1: A simple polygon on the lattice points.

4. Let n be a natural number ($n \geq 0$) and p be a prime ($p \geq 2$). Let s be the sum of the p-ary digits in the representation of n in base p. Let m be the multiplicity of the factor p in $n!$, i.e., the maximum value m such that p^m divides $n!$. For example, if $n = 6$ and $p = 2$, then $n = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 110_2$ and hence $s = 1 + 1 + 0 = 2$. Moreover, 2^4 divides $6!$ but 2^5 does not and therefore $m = 4$.

   Prove that
$$m = \frac{n-s}{p-1} \; .$$

   In the example above, $\frac{n-s}{p-1} = \frac{6-2}{2-1} = 4 = m$.

5. Consider the following algorithm for computing the square of the input number n, which is assumed to be a positive integer.

```
Algorithm mySquare(n);
begin
    // assume that n > 0
    x := n;
    y := 0;
    while x > 0 do
        y := y + 2 × x − 1;
        x := x − 1
    od;
    mySquare := y
end
```

   State a suitable loop invariant for the while loop and prove its correctness. The loop invariant should be strong enough for deducing that, when the while loop terminates, the value of y equals the square of n.