Algorithms 2024: Analysis of Algorithms

(Based on [Manber 1989])

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1 Introduction

Introduction

- The purpose of algorithm analysis is to predict the behavior (running time, space requirement, etc.) of an algorithm *without implementing it* on a specific computer. (Why?)
- As the exact behavior of an algorithm is hard to predict, the analysis is usually an *approximation*:
 - Relative to the input size (usually denoted by n): input possibilities too enormous to elaborate
 - Asymptotic: should care more about larger inputs
 - Worst-Case: easier to do, often representative (Why not average-case?)
- Such an approximation is usually good enough for comparing different algorithms for the same problem.

Complexity

- Theoretically, "complexity of an algorithm" is a more precise term for "approximate behavior of an algorithm".
- Two most important measures of complexity:
 - Time Complexity an upper bound on the number of steps that the algorithm performs.
 - Space Complexity an upper bound on the amount of temporary storage required for running the algorithm (excluding the input, the output, and the program itself).
- We will focus on time complexity.

Comparing Running Times

- How do we compare the following running times?
 - $1. \,\, 100n$
 - 2. $2n^2 + 50$
 - 3. $100n^{1.8}$
- We will study an approach (the *O* notation) that allows us to ignore constant factors and concentrate on the behavior as *n* goes to infinity.
- For most algorithms, the constants in the expressions of their running times tend to be small.

2 The O Notation

The O Notation

- A function g(n) is O(f(n)) for another function f(n) if there exist constants c and N such that, for all $n \ge N$, $g(n) \le cf(n)$.
- The function g(n) may be substantially less than cf(n); the O notation bounds it only from above.
- The O notation allows us to ignore constants conveniently.
- Examples:
 - $-5n^2 + 15 = O(n^2)$. (cf. $5n^2 + 15 \le O(n^2)$ or $5n^2 + 15 \in O(n^2)$)
 - $-5n^2 + 15 = O(n^3)$. (cf. $5n^2 + 15 \le O(n^3)$ or $5n^2 + 15 \in O(n^3)$)
 - As part of an expression like $T(n) = 3n^2 + O(n)$.

The *O* Notation (cont.)

• No need to specify the base of a logarithm.

$$-\log_2 n = \frac{\log_{10} n}{\log_{10} 2} = \frac{1}{\log_{10} 2} \log_{10} n.$$

- For example, we can just write $O(\log n)$.
- O(1) denotes a constant.

Properties of O

• We can add and multiply with O.

Lemma 1 (3.2). 1. If f(n) = O(s(n)) and g(n) = O(r(n)), then f(n) + g(n) = O(s(n) + r(n)). 2. If f(n) = O(s(n)) and g(n) = O(r(n)), then $f(n) \cdot g(n) = O(s(n) \cdot r(n))$.

/* There exist constants c_1 , N_1 , c_2 , and N_2 such that, for all $n \ge N_1$, $f(n) \le c_1 s(n)$ and, for all $n \ge N_2$, $g(n) \le c_2 r(n)$. Assume without loss of generality that $c_1 \ge c_2$ and $N_1 \ge N_2$. Then, for all $n \ge N_1$, $f(n) + g(n) \le c_1 s(n) + c_2 r(n) \le c_1 s(n) + c_1 r(n) = c_1 (s(n) + r(n))$, i.e., f(n) + g(n) = O(s(n) + r(n)). Also, for all $n \ge N_1$, $f(n) \cdot g(n) \le c_1 s(n) \cdot c_2 r(n) = c_1 c_2 (s(n) \cdot r(n))$, which implies that there exist constants c and N such that, for all $n \ge N$, $f(n) \cdot g(n) \le c(s(n) \cdot r(n))$, i.e., $f(n) \cdot g(n) = O(s(n) \cdot r(n))$.

• However, we cannot subtract or divide with O.

$$-2n = O(n), n = O(n), \text{ and } 2n - n = n \neq O(n - n).$$

- $n^2 = O(n^2), n = O(n^2), \text{ and } n^2/n = n \neq O(n^2/n^2).$

3 Speed of Growth

Polynomial vs. Exponential

- A function f(n) is monotonically growing (or monotonically increasing) if $n_1 \ge n_2$ implies that $f(n_1) \ge f(n_2)$.
- An exponential function grows at least as fast as a polynomial function does.

Theorem 2 (3.1). For all constants c > 0 and a > 1, and for all monotonically growing functions $f(n), (f(n))^c = O(a^{f(n)}).$

- Examples:
 - Take n as f(n), $n^c = O(a^n)$.

- Take $\log_a n$ as f(n), $(\log_a n)^c = O(a^{\log_a n}) = O(n)$.

Speed of Growth

| $\log n$ | n | $n \log n$ | n^2 | n^3 | 2^n |
|----------|----|------------|-------|--------|-----------------------|
| 0 | 1 | 0 | 1 | 1 | 2 |
| 1 | 2 | 2 | 4 | 8 | 4 |
| 2 | 4 | 8 | 16 | 64 | 16 |
| 3 | 8 | 24 | 64 | 512 | 256 |
| 4 | 16 | 64 | 256 | 4,096 | $65,\!536$ |
| 5 | 32 | 160 | 1,024 | 32,768 | $4,\!294,\!967,\!296$ |

Table: Function values.

Source: redrawn from [E. Horowitz et al. 1998, Table 1.7].

Speed of Growth (cont.)

| | $time_1$ | $time_2$ | $time_3$ | $time_4$ |
|---------------|-----------------|-----------------|-----------------|-----------------|
| running times | 1000 steps/sec | 2000 steps/sec | 4000 steps/sec | 8000 steps/sec |
| $\log n$ | 0.010 | 0.005 | 0.003 | 0.001 |
| n | 1 | 0.5 | 0.25 | 0.125 |
| $n \log n$ | 10 | 5 | 2.5 | 1.25 |
| $n^{1.5}$ | 32 | 16 | 8 | 4 |
| n^2 | 1000 | 500 | 250 | 125 |
| n^3 | 1,000,000 | 500,000 | 250,000 | 125,000 |
| 1.1^{n} | 10^{39} | 10^{39} | 10^{38} | 10^{38} |

Table: Running times (in seconds) under different assumptions (n = 1000).

Source: redrawn from [Manber 1989, Table 3.1].

$O, o, \Omega, \text{ and } \Theta$

- Let T(n) be the number of steps required to solve a given problem for input size n.
- We say that $T(n) = \Omega(g(n))$ or the problem has a lower bound of $\Omega(g(n))$ if there exist constants c and N such that, for all $n \ge N$, $T(n) \ge cg(n)$.
- If a certain function f(n) satisfies both f(n) = O(g(n)) and $f(n) = \Omega(g(n))$, then we say that $f(n) = \Theta(g(n))$.
- We say that f(n) = o(g(n)) if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

Polynomial vs. Exponential (cont.)

• An exponential function grows *faster* than a polynomial function does.

Theorem 3 (3.3). For all constants c > 0 and a > 1, and for all monotonically growing functions f(n), we have

$$(f(n))^c = o(a^{f(n)}).$$

• Consider a previous example again: Take $\log_a n$ as f(n). For all c > 0 and a > 1,

$$(\log_a n)^c = o(a^{\log_a n}) = o(n)$$

4 Sums

 \mathbf{Sums}

- Techniques for summing expressions are essential for complexity analysis.
- For example, given that we know

$$S_0(n) = \sum_{i=1}^n 1 = n$$

and

$$S_1(n) = \sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2},$$

we want to compute the sum

$$S_2(n) = \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

Sums (cont.)

From

$$(i+1)^3 = i^3 + 3i^2 + 3i + 1,$$

we have

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1.$$

$$2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

$$3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$$

$$4^3 - 3^3 = 3 \times 3^2 + 3 \times 3 + 1$$

$$\dots \dots \dots$$

$$(n+1)^3 - n^3 = 3 \times n^2 + 3 \times n + 1$$

$$(n+1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n)$$

$$(S_3(n+1) - S_3(1)) - S_3(n) = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n)$$

Sums (cont.)

• So, we have

$$(n+1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n).$$

- Given $S_0(n)$ and $S_1(n)$, the sum $S_2(n)$ can be computed by straightforward algebra.
- Recall that the left-hand side $(n + 1)^3 1$ equals $(S_3(n + 1) S_3(1)) S_3(n)$, a result from "shifting and canceling" terms of two sums.
- This generalizes to $S_k(n)$, for k > 2.
- Similar shifting and canceling techniques apply to other kinds of sums.

/* We actually will need to obtain an upper bound for the sum of n upper bounds. For instance, $\sum_{i=1}^{n} O(1) = O(\sum_{i=1}^{n} 1) = O(n)$, $\sum_{i=1}^{n} O(i) = O(\sum_{i=1}^{n} i) = O(\frac{n(n+1)}{2}) = O(n^2)$, etc. */

5 Recurrence Relations

Recurrence Relations

- A recurrence relation is a way to define a function by an expression involving the same function.
- The Fibonacci numbers, for example, can be defined as follows:

$$\begin{cases}
F(1) = 1 \\
F(2) = 1 \\
F(n) = F(n-2) + F(n-1)
\end{cases}$$

We would need k-2 steps to compute F(k).

- It is more convenient to have an explicit (or closed-form) expression.
- To obtain the explicit expression is called *solving* the recurrence relation.

Guessing and Proving an Upper Bound

- Recurrence relation: $\left\{ \begin{array}{l} T(2)=1\\ T(2n)\leq 2T(n)+2n-1 \end{array} \right.$
- Guess: $T(n) = O(n \log n)$.

/* It suffices to show that $T(n) \le n \log n$, for every $n \ge 2$ (that is a power of 2), i.e., let c be 1 and N be 2 in the definition of O.*/

- Proof:
 - 1. Base case: $T(2) \leq 2 \log 2$.
 - 2. Inductive step:

$$\begin{array}{rcl} T(2n) & \leq & 2T(n) + 2n - 1 \\ & \leq & 2(n\log n) + 2n - 1 \\ & = & 2n\log n + 2n\log 2 - 1 \\ & \leq & 2n(\log n + \log 2) \\ & = & 2n\log 2n \end{array}$$

Solving the Fibonacci Relation

- We will study two techniques for solving the Fibonacci relation.
 - 1. One uses the characteristic equation
 - 2. The other uses generating functions
- These techniques may be generalized to handle recurrence relations of the form

$$F(n) = b_1 F(n-1) + b_2 F(n-2) + \dots + b_k F(n-k)$$

for a constant k.

Using the Characteristic Equation

- F(n) nearly doubles every time and should be an exponential function.
- But what is the base of the exponential function?
- The base a should satisfy $a^n = a^{n-1} + a^{n-2}$, which implies $a^2 = a + 1$ (called the characteristic equation).
- There are two solutions to the characteristic equation: $a_1 = \frac{1+\sqrt{5}}{2}$ and $a_2 = \frac{1-\sqrt{5}}{2}$.
- Any linear combination of a_1^n and a_2^n solves the recurrence relation.

Using the Characteristic Equation (cont.)

• So, the general solution is

$$c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n.$$

• To fit the values of F(1) and F(2), c_1 and c_2 must satisfy

$$c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1$$

$$c_1(\frac{1+\sqrt{5}}{2})^2 + c_2(\frac{1-\sqrt{5}}{2})^2 = 1$$

• Therefore, $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$, and the exact solution to the Fibonacci relation is

$$F(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using Generating Functions

- *Generating functions* provide a systematic, effective means for representing and manipulating infinite sequences (of numbers).
- We use them here to derive a closed-form representation of the Fibonacci numbers.
- Below are a few basic generating functions:

| gen. | power series | generated sequence |
|------------------|---|-------------------------------------|
| func. | | |
| $\frac{1}{1-z}$ | $1+z+z^2+\cdots+z^n+\cdots$ | $1, 1, 1, \cdots, 1, \cdots$ |
| $\frac{c}{1-az}$ | $c + caz + ca^2 z^2 + \dots + ca^n z^n + \dots$ | $c, ca, ca^2, \cdots, ca^n, \cdots$ |
| $\frac{1}{1+z}$ | $1-z+z^2+\cdots+(-1)^nz^n+\cdots$ | $1, -1, 1, \cdots, (-1)^n, \cdots$ |

The fraction $\frac{1}{1+z}$ can be seen as $\frac{1}{1-(-z)}$.

• The second one is a generalization of the first and will be used in our solution.

Using Generating Functions (cont.)

Let $G(z) = 0 + F_1 z + F_2 z^2 + F_3 z^3 + \dots + F_n z^n + \dots$ (a generating function for the Fibonacci numbers; F(n) is written as F_n here).

$$G(z) = F_{1}z + F_{2}z^{2} + F_{3}z^{3} + \dots + F_{n}z^{n} + F_{n+1}z^{n+1} + \dots$$

$$zG(z) = F_{1}z^{2} + F_{2}z^{3} + \dots + F_{n-1}z^{n} + F_{n}z^{n+1} + \dots$$

$$z^{2}G(z) = F_{1}z^{3} + F_{2}z^{4} + \dots + F_{n-2}z^{n} + F_{n-1}z^{n+1} + \dots$$

$$(1 - z - z^{2})G(z) = z$$

$$G(z) = \frac{z}{1-z-z^2} \left(= \frac{z}{(1-\frac{1+\sqrt{5}}{2}z)(1-\frac{1-\sqrt{5}}{2}z)} \right)$$
$$= \frac{\frac{1}{\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} + \frac{-\frac{1}{\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z}$$

/*

*/

$$\begin{aligned} G(z) &= \frac{\frac{1}{\sqrt{5}}}{1 - \frac{1 + \sqrt{5}}{2}z} + \frac{-\frac{1}{\sqrt{5}}}{1 - \frac{1 - \sqrt{5}}{2}z} \\ &= (\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\frac{1 + \sqrt{5}}{2}z + \frac{1}{\sqrt{5}}(\frac{1 + \sqrt{5}}{2})^2 z^2 + \dots + \frac{1}{\sqrt{5}}(\frac{1 + \sqrt{5}}{2})^n z^n + \dots) + \\ &\quad (-\frac{1}{\sqrt{5}} + (-\frac{1}{\sqrt{5}})\frac{1 - \sqrt{5}}{2}z + (-\frac{1}{\sqrt{5}})(\frac{1 - \sqrt{5}}{2})^2 z^2 + \dots + (-\frac{1}{\sqrt{5}})(\frac{1 - \sqrt{5}}{2})^n z^n + \dots) \\ &= z + z^2 + \dots + (\frac{1}{\sqrt{5}}(\frac{1 + \sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1 - \sqrt{5}}{2})^n)z^n + \dots \end{aligned}$$

Therefore, $F_n = \frac{1}{\sqrt{5}} (\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}} (\frac{1-\sqrt{5}}{2})^n$.

6 Divide and Conquer Relations

Divide and Conquer Relations

• The running time T(n) of a divide-and-conquer algorithm satisfies

$$T(n) = aT(n/b) + O(n^k)$$

where

- -a is the number of subproblems,
- $-\ n/b$ is the size of each subproblem, and
- $-O(n^k)$ is the time spent on dividing the problem and combining the solutions.

Divide and Conquer Relations (cont.)

Assume, for simplicity, $n = b^m$ $(\frac{n}{b^m} = 1, \frac{n}{b^{m-1}} = b$, etc.).

$$T(n) = aT(\frac{n}{b}) + O(n^k)$$

= $a(aT(\frac{n}{b^2}) + O((\frac{n}{b})^k)) + O(n^k)$
= $a(a(aT(\frac{n}{b^3}) + O((\frac{n}{b^2})^k)) + O((\frac{n}{b})^k)) + O(n^k)$
...
= $a(a(\dots(aT(\frac{n}{b^m}) + O((\frac{n}{b^{m-1}})^k)) + \dots) + O((\frac{n}{b})^k)) + O(n^k)$

Assuming T(1) = O(1) (and recalling $n = b^m$, i.e., $m = \log_b n$),

$$T(n) = a^m \times O(1) + \sum_{i=1}^m a^{m-i}O(b^{ik}) = O(a^m) + a^m \sum_{i=1}^m O((\frac{b^k}{a})^i).$$

Divide and Conquer Relations (cont.)

As $m = \log_b n$ and $a^m = a^{\log_b n} = n^{\log_b a}$.

$$T(n) = O(n^{\log_b a}) + O(n^{\log_b a}) \times O(\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i).$$

- $O(n^{\log_b a})$ is the accumulative time for computing all the subproblems.
- $O(n^{\log_b a}) \times O(\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i)$ is the accumulative time for dividing problems and combining solutions.
- Three cases to consider: $\frac{b^k}{a} < 1$, $\frac{b^k}{a} = 1$, and $\frac{b^k}{a} > 1$.

/* Case 1: $\frac{b^k}{a} < 1$. The geometric series $\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i$ converges to some constant. So, $T(n) = O(n^{\log_b a}) + O(n^{\log_b a}) \times O(1) = O(n^{\log_b a})$.

Case 2: $\frac{b^k}{a} = 1$, i.e., $\log_b a = k$. $O(\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i) = O(\log_b n) = O(\log_b n)$. So, $T(n) = O(n^{\log_b a}) + O(n^{\log_b a}) \times O(\log n) = O(n^k \log n)$.

 $\begin{array}{l} \text{Case 3:} \quad \frac{b^{k}}{a} > 1. \quad O(\sum_{i=1}^{\log_{b} n} (\frac{b^{k}}{a})^{i}) = O(\frac{b^{k}}{a} \frac{(\frac{b^{k}}{a})^{\log_{b} n} - 1}{\frac{b^{k}}{a} - 1}) = O((\frac{b^{k}}{a})^{\log_{b} n}) = O(\frac{(b^{k})^{\log_{b} n}}{a^{\log_{b} n}}) = O(\frac{(b^{\log_{b} n})^{k}}{a^{\log_{b} n}}) =$

Divide and Conquer Relations (cont.)

Theorem 4 (3.4). The solution of the recurrence relation $T(n) = aT(n/b) + O(n^k)$, where a and b are integer constants, $a \ge 1$, $b \ge 2$, and k is a non-negative real constant, is

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > b^k \\ O(n^k \log n) & \text{if } a = b^k \\ O(n^k) & \text{if } a < b^k \end{cases}$$

This theorem may be slightly generalized by replacing $O(n^k)$ with some f(n), but the current form is sufficient for the cases we will encounter. Due to its generality and usefulness, the theorem has conventionally been referred to as "the master theorem".

/* Example 1: Suppose T(n) = T(n/2) + O(1) (arising from, e.g., binary search). In this case, a = 1, b = 2, and k = 0. We have $a = b^k$ and the second case of the theorem applies. Therefore, $T(n) = O(n^0 \log n) = O(\log n)$.

Example 2: Suppose T(n) = 2T(n/2) + O(n) (arising from, e.g., merge sort). In this case, a = 2, b = 2, and k = 1. We have $a = b^k$ and again the second case of the theorem applies. Therefore, $T(n) = O(n \log n)$.

Recurrent Relations with Full History

• Example One:

$$T(n) = c + \sum_{i=1}^{n-1} T(i),$$

where c is a constant and T(1) is given separately.

• $T(n) - T(n-1) = (c + \sum_{i=1}^{n-1} T(i)) - (c + \sum_{i=1}^{n-2} T(i)) = T(n-1)$; hence, T(n) = 2T(n-1). (This holds only for $n \ge 3$.) /* The relation T(n) = 2T(n-1) does not hold for n = 2, as T(2) - T(1) = c (not T(1)). */

• So, we get

$$\begin{cases} T(2) = c + T(1) \\ T(n) = 2T(n-1) & \text{if } n \ge 3 \end{cases}$$

which is easier to solve.

• $T(n+1) = (c+T(1))2^{n-1}$, for $n \ge 2$.

Recurrent Relations with Full History (cont.)

• Example Two:

$$T(n) = n + 1 + \frac{2}{n} \sum_{i=1}^{n-1} T(i), \text{ for } n \ge 2.$$
 $T(1) = 0.$

• Multiply both sides of the equation with n for T(n) and (n+1) for T(n+1).

$$nT(n) = n(n+1) + 2\sum_{i=1}^{n-1} T(i)$$

(n+1)T(n+1) = (n+1)(n+2) + 2\sum_{i=1}^{n} T(i)

• Take the difference.

$$(n+1)T(n+1) - nT(n) = 2n + 2 + 2T(n)$$

which implies

$$T(n+1) = \frac{n+2}{n+1}T(n) + 2$$

Recurrent Relations with Full History (cont.)

• So, we have

$$T(n+1) = \frac{n+2}{n+1}T(n) + 2$$
, for $n \ge 2$.

• Expanding and canceling.

$$\begin{split} T(n) \\ &= 2 + \frac{n+1}{n} \left(2 + \frac{n}{n-1} \left(2 + \frac{n-1}{n-2} \left(\cdots \left(2 + \frac{4}{3} T(2) \right) \cdots \right) \right) \right) \\ &= 2 \left(1 + \frac{n+1}{n} + \frac{n+1}{n} \frac{n}{n-1} + \frac{n+1}{n} \frac{n}{n-1} \frac{n-1}{n-2} + \cdots + \left(\frac{n+1}{n} \frac{n}{n-1} \cdots \frac{4}{3} \frac{3}{2} \right) \right) \\ &= 2 (n+1) \left(\frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{3} + \frac{1}{2} \right) \\ &\leq 2 + 2 (n+1) \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1 \right) \\ &= O(n \log n) \end{split}$$

(Note: T(1) = 0 and $T(2) = (2+1) + \frac{2}{2}T(1) = 3$)

7 Useful Facts

Useful Facts

• Bounding a summation by an integral: If f(x) is monotonically *increasing*, then

$$\sum_{i=1}^n f(i) \le \int_1^{n+1} f(x) dx.$$

If f(x) is monotonically decreasing, then

$$\sum_{i=1}^n f(i) \le f(1) + \int_1^n f(x) dx.$$

• Stirling's approximation

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O(1/n)\right).$$

Bounding a Summation by an Integral



/* This technique can be used to show that $\int_0^n f(x)dx \leq \sum_{i=1}^n f(i)$, by shifting the *n* vertical bars (which represent $\sum_{i=1}^n f(i)$) in the diagram to the left by one unit. When f(x) is monotonically decreasing, we state that $\sum_{i=1}^n f(i) \leq f(1) + \int_1^n f(x)dx$, rather than $\sum_{i=1}^n f(i) \leq \int_0^n f(x)dx$, as the part $\int_0^1 f(x)dx$ might go to infinity and would not be a good upper bound. Isolating the first term of the sum, we have $\sum_{i=1}^n f(i) = f(1) + \sum_{i=2}^n f(i) \leq f(1) + \int_1^n f(x)dx$. It can also be shown that $\int_1^{n+1} f(x)dx \leq \sum_{i=1}^n f(i)$. */

Useful Facts (cont.)

• Harmonic series

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + O(1/n),$$

where $\gamma = 0.577...$ is Euler's constant. So, $H_n = O(\log n)$.

/* The upper bound may also be obtained using an integral. $\sum_{k=1}^{n} \frac{1}{k} \leq \frac{1}{1} + \int_{1}^{n} \frac{1}{x} dx = 1 + \ln n =$ $O(\ln n) = O(\log n). */$

• Sum of logarithms

$$\sum_{i=1}^{n} \lfloor \log_2 i \rfloor = (n+1) \lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2$$
$$= \Theta(n \log n).$$

/* The value of $\lfloor x \rfloor$ (the floor of x) is the largest integer that is less than or equal to x; or equivalently, $\lfloor x \rfloor = m$ if and only if $m \le x < m + 1$ for some integer m. */

 $/ \sum_{i=1}^{n} \lfloor \log_2 i \rfloor \leq \sum_{i=1}^{n} \log_2 i = \log_2(n!) = \log_2(\sqrt{2\pi n}(\frac{n}{e})^n (1 + O(1/n))) = O(\log_2(\sqrt{2\pi n}(\frac{n}{e})^n)) = O(\log_2\sqrt{2\pi n} + \log_2(\frac{n}{e})^n) = O(\log_2\sqrt{2\pi n} + n\log_2(\frac{n}{e})) = O(n\log n).$ The other direction $\sum_{i=1}^{n} \lfloor \log_2 i \rfloor \geq (\sum_{i=1}^{n} \log_2 i) - n. * /$