# Data Structures

## A Supplement
## (Based on [Manber 1989])

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# Heaps

- A (max binary) heap is a complete binary tree whose keys satisfy the heap property:
  *the key of every node is greater than or equal to the key of any of its children*.
- It supports the two basic operations of a priority queue:

# Heaps

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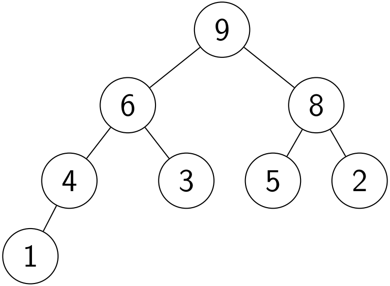
  *the key of every node is greater than or equal to the key of any of its children*.
- It supports the two basic operations of a priority queue:
  - *Insert*($x$): insert the key $x$ into the heap.
  - *Remove*(): remove and return the largest key from the heap.

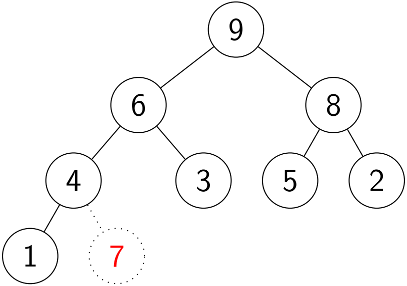
# Heaps (cont.)

- A complete binary tree can be represented implicitly by an array $A$ as follows:
  1. The root is stored in $A[1]$.
  2. The left child of $A[i]$ is stored in $A[2i]$ and the right child is stored in $A[2i+1]$.

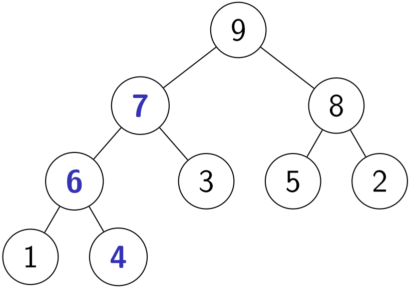



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 6 | 8 | 4 | 3 | 5 | 2 | 1 | _ | _ | _ | _ |

# Heaps (cont.)




Before *Insert*(7)

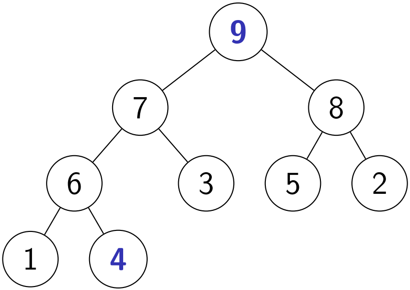



After *Insert*(7)

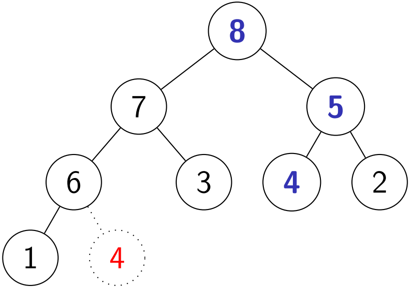
# Heaps (cont.)

```
Algorithm Insert_to_Heap (A, n, x);
begin
        n := n + 1;
        A[n] := x;
        child := n;
        parent := n div 2;
        while parent ≥ 1 do
              if A[parent] < A[child] then
                 swap(A[parent], A[child]);
                 child := parent;
                 parent := parent div 2
              else parent := 0
end
```

# Heaps (cont.)




Before *Remove*()




After *Remove*()

# Heaps (cont.)

**Algorithm Remove_Max_from_Heap** $(A, n)$;
**begin**
  **if** $n = 0$ **then** print "the heap is empty"
  **else** $Top\_of\_the\_Heap := A[1]$;
    $A[1] := A[n]$; $n := n - 1$;
    $parent := 1$; $child := 2$;
    **while** $child \leq n$ **do**
      **if** $child + 1 \leq n$ and $A[child] < A[child + 1]$ **then**
        $child := child + 1$;
      **if** $A[child] > A[parent]$ **then**
        $swap(A[parent], A[child])$;
        $parent := child$;
        $child := 2 * child$
      **else** $child := n + 1$
**end**

# AVL Trees

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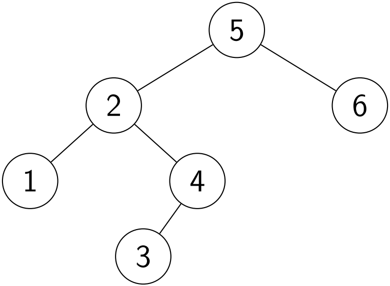
## Definition

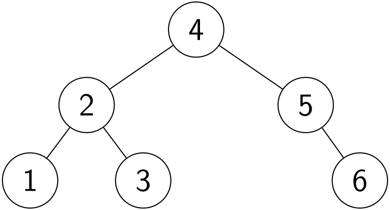
An AVL tree is a binary search tree such that, for every node, the difference between the heights of its left and right subtrees is at most 1 (the height of an empty tree is defined as 0).

This definition guarantees a maximal height of $O(\log n)$ for any AVL tree of n nodes.

# AVL Trees (cont.)




A binary search tree
but NOT an AVL tree




A binary search tree
and also an AVL tree

# AVL Trees (cont.)

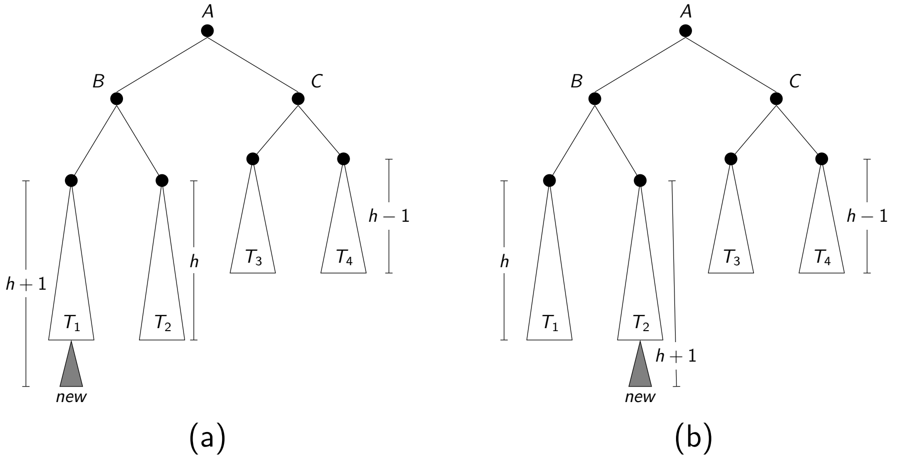
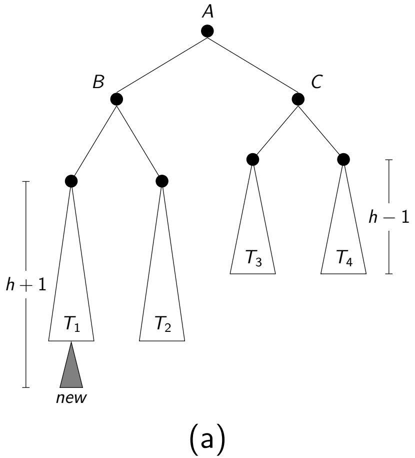



Figure: Insertions that invalidate the AVL property. Note that this tree rooted at $A$ shown here may be part of a larger AVL tree.

Source: redrawn from [Manber 1989, Figure 4.13].

# AVL Trees (cont.)



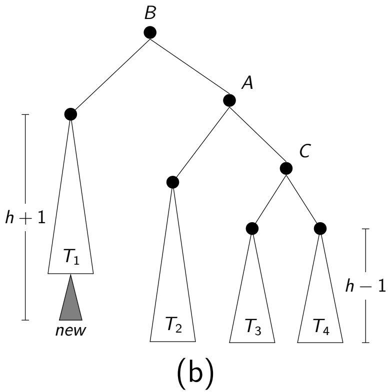




Figure: A single rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.14].

# AVL Trees (cont.)

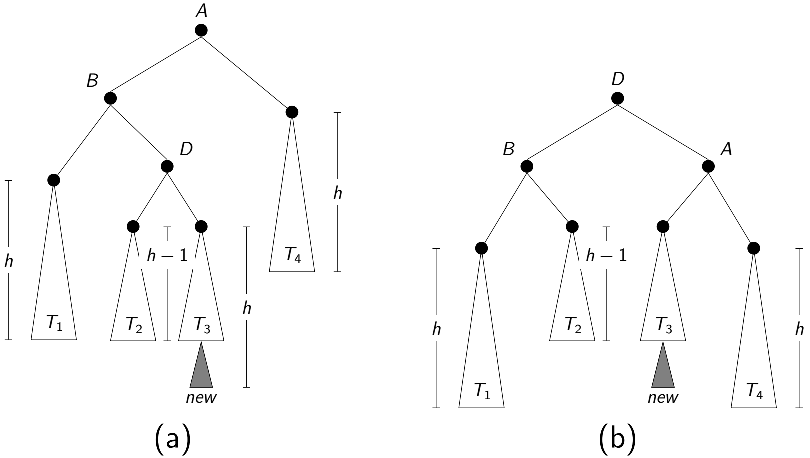



Figure: A double rotation: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.15].

# Union-Find

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- There are $n$ elements $x_1, x_2, \cdots, x_n$ divided into groups. Initially, each element is in a group by itself.
- Two operations on the elements and groups:
  - *find*$(A)$: returns the name of $A$'s group.
  - *union*$(A, B)$: combines $A$'s and $B$'s groups to form a new group with a unique name.
- To tell if two elements are in the same group, one may issue a find operation for each element and see if the returned names are the same.

# Union-Find (cont.)

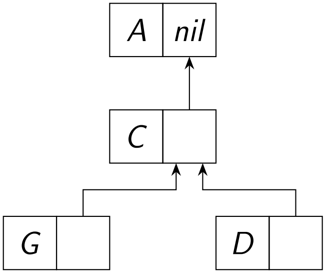


Figure: The representation for the union-find problem.

Source: redrawn from [Manber 1989, Figure 4.16].

# Balancing

- The root also stores the number of elements in (i.e., the size of) its group.
- To *balance* the tree resulted from a union operation, *let the smaller group join the larger group* and update the size of the larger group accordingly.

**Theorem (Theorem 4.2)**

*If balancing is used, then any tree of height $h$ ($\geq 0$) must contain at least $2^h$ elements.*

- Any sequence of $m$ find or union operations (where $m \geq n$) takes $O(m \log n)$ steps.

# Union-Find (cont.)

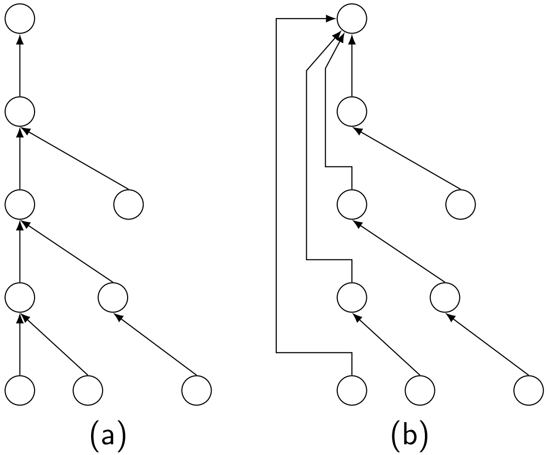


Figure: Path compression: (a) before; (b) after.

Source: redrawn from [Manber 1989, Figure 4.17].

# Effect of Path Compression

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**Theorem (Theorem 4.3)**

*If both balancing and path compression are used, any sequence of $m$ find or union operations (where $m \geq n$) takes $O(m \log^* n)$ steps.*

The value of $\log^* n$ intuitively equals the number of times that one has to apply log to $n$ to bring its value down to 1.

# Code for Union-Find

```
Algorithm Union_Find_Init(A,n);
begin
  for i := 1 to n do
      A[i].parent := nil;
      A[i].size := 1
end

Algorithm Find(a);
begin
  if A[a].parent <> nil then
     A[a].parent := Find(A[a].parent);
     Find := A[a].parent;
  else
     Find := a
end
```

# Code for Union-Find (cont.)

```
Algorithm Union(a,b);
begin
  x := Find(a);
  y := Find(b);
  if x <> y then
     if A[x].size > A[y].size then
        A[y].parent := x;
        A[x].size := A[x].size + A[y].size;
     else
        A[x].parent := y;
        A[y].size := A[y].size + A[x].size
end
```