# Binary Decision Diagrams

## (Based on [Clarke *et al.* 1999] and [Bryant 1986])

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# Boolean Functions

- Boolean functions are widely used in
  - digital logic design,
  - testing,
  - artificial intelligence, and
  - model checking.
- Boolean operators
  - And: $x \cdot y$ $(x \wedge y)$
  - Or: $x + y$ $(x \vee y)$
  - Not: $\bar{x}$ $(\neg x)$
  - If and only if: $\leftrightarrow$
- Example: $f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \cdot (x_3 \leftrightarrow x_4)$

# Representations of Boolean Functions

- A variety of methods have been developed for representing and manipulating Boolean functions such as:
    - Karnaugh map
    - Sum-of-products form
    - Truth table
    - Binary decision tree
- But these representations are quite impractical, because every function of $n$ arguments has a representation of size $2^n$ or more.

# Karnaugh Map

A Karnaugh table for $f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \cdot (x_3 \leftrightarrow x_4)$.

| $x_3x_4$<br>$x_1x_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 1 | 0 |
| 01 | 0 | 0 | 0 | 0 |
| 11 | 1 | 0 | 1 | 0 |
| 10 | 0 | 0 | 0 | 0 |

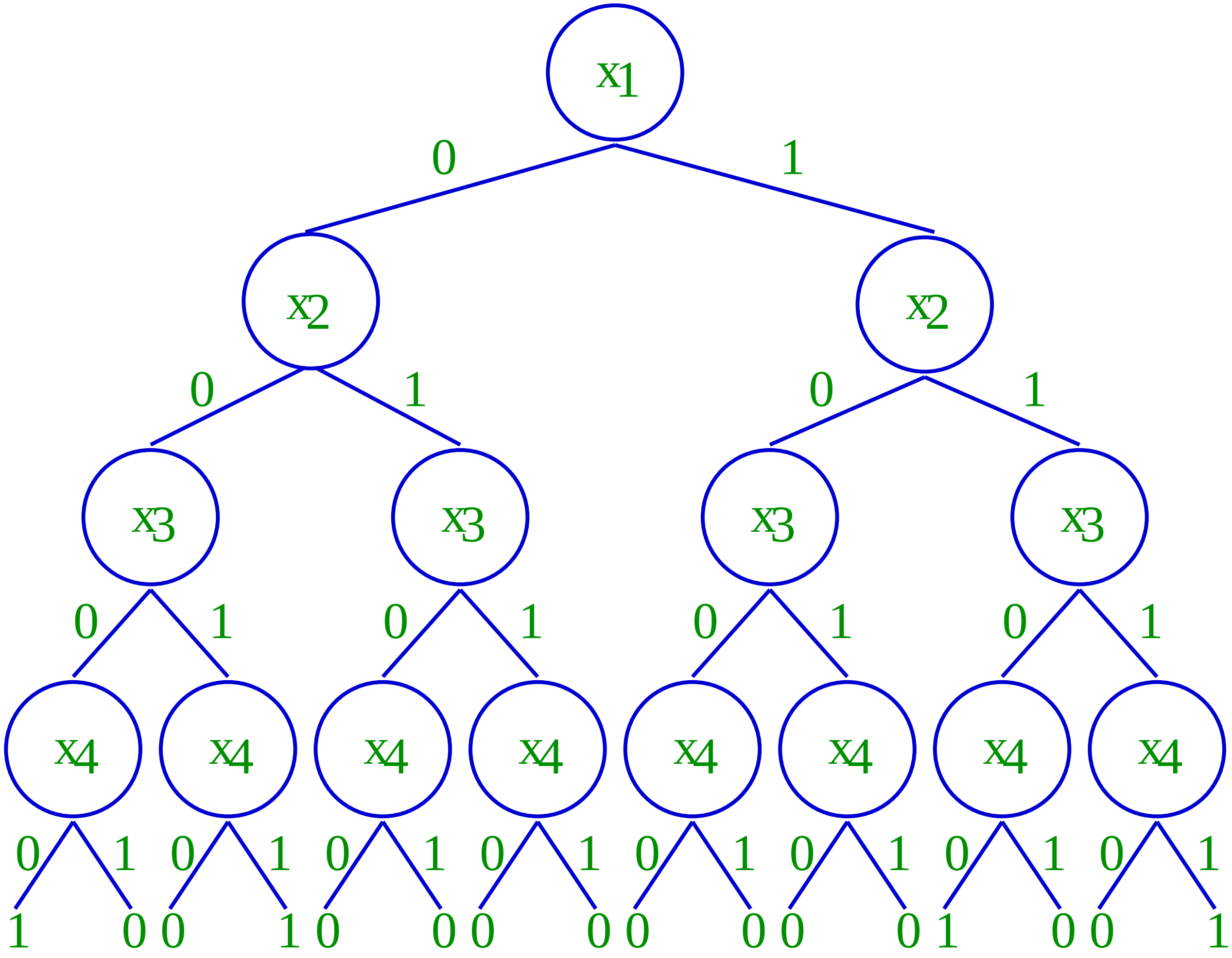
# Truth Table

A truth table for $f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \cdot (x_3 \leftrightarrow x_4)$.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $f$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

# Binary Decision Tree

A binary decision tree for
$f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \cdot (x_3 \leftrightarrow x_4)$.



# Representations of Boolean Functions (cont.)

- More practical approaches utilize representations that, at least for many functions, are not of exponential size.
  - reduced sum of products
  - factored into unate functions
- But these representations still suffer from several drawbacks:
  - Certain common functions require representations of exponential size.
  - Performing a simple operation could yield a function with an exponential representation.
  - None of these representations are *canonical forms*.

# Binary Decision Diagrams

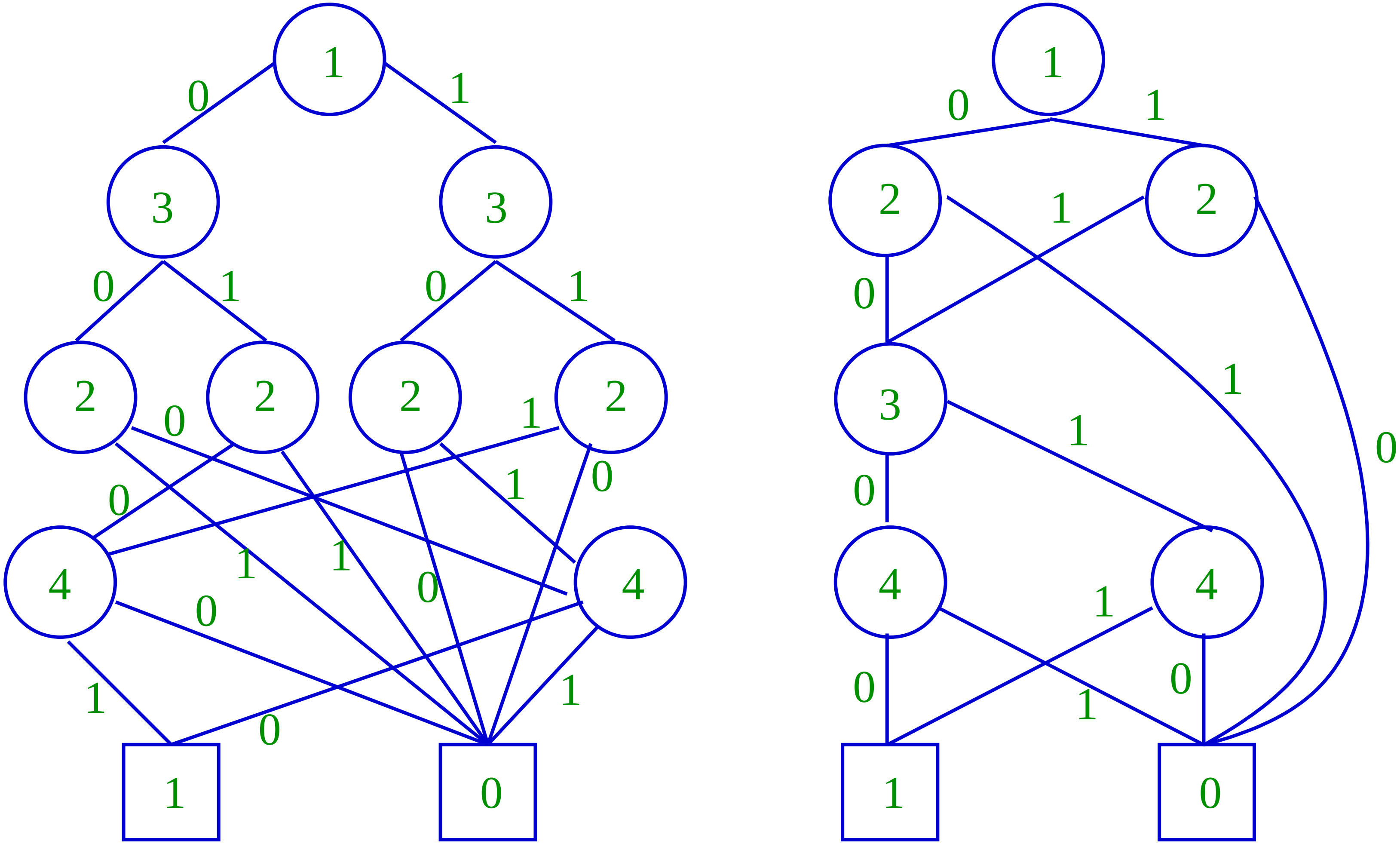
- A binary decision diagram (BDD) represents a Boolean function as a rooted, directed acyclic graph (function graph).
- We use $r(G)$ to denote the root of a function graph $G$.
- The vertex set $V$ of a function graph $G$ contains two types of vertices.
  - A nonterminal vertex $v$ has
    - an argument index $index(v) \in \{1, \ldots, n\}$ and
    - two children $low(v), high(v) \in V$.
  - A terminal vertex $v$ has a value $value(v) \in \{0, 1\}$

# Ordered Binary Decision Diagrams

- An ordered binary decision diagram (ODBB) is defined by imposing a total ordering over the nonterminal vertices.
  - For any nonterminal vertex $v$,
    - if $low(v)$ is nonterminal, then we must have $index(v) < index(low(v))$;
    - if $high(v)$ is nonterminal, then we must have $index(v) < index(high(v))$.
- Further minimality conditions will be introduced later.
- OBDDs are representations of Boolean functions with *canonical forms* and *reasonable size*.
- The size of the graph is highly sensitive to arguments ordering.

# Ordering

Two OBDDs for $f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \cdot (x_3 \leftrightarrow x_4)$ with different orderings.



# Notations

- All functions have the same $n$ arguments: $x_1, \cdots, x_n$.
- A restriction of $f$ is denoted $f|_{x_i=b}$ where $b$ is a constant.

$$f|_{x_i=b}\,(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$$

- A composition of $f$ and $g$ is denoted $f|_{x_i=g}$ where $g$ is a Boolean function.

$$f|_{x_i=g}\,(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n)$$

# Notations (cont.)

- The Shannon expansion of a function around variable $x_i$ is given by:

$$f = x_i \cdot f\,|_{x_i=1} + \bar{x}_i \cdot f\,|_{x_i=0}$$

- The dependency set of a function $f$ is denoted $I_f$.

$$I_f = \{i \mid f\,|_{x_i=0} \neq f\,|_{x_i=1}\}$$

- The satisfying set of a function $f$ is denoted $S_f$.

$$S_f = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 1\}$$

# Correspondence

- A function graph (OBDD) $G$ having root vertex $v$ denotes a function $f_v$ defined recursively as follows:
  - If $v$ is a terminal vertex:
    - If $value(v) = 1$, then $f_v = 1$.
    - If $value(v) = 0$, then $f_v = 0$.
  - If $v$ is a nonterminal vertex with $index(v) = i$, then $f_v$ is the function

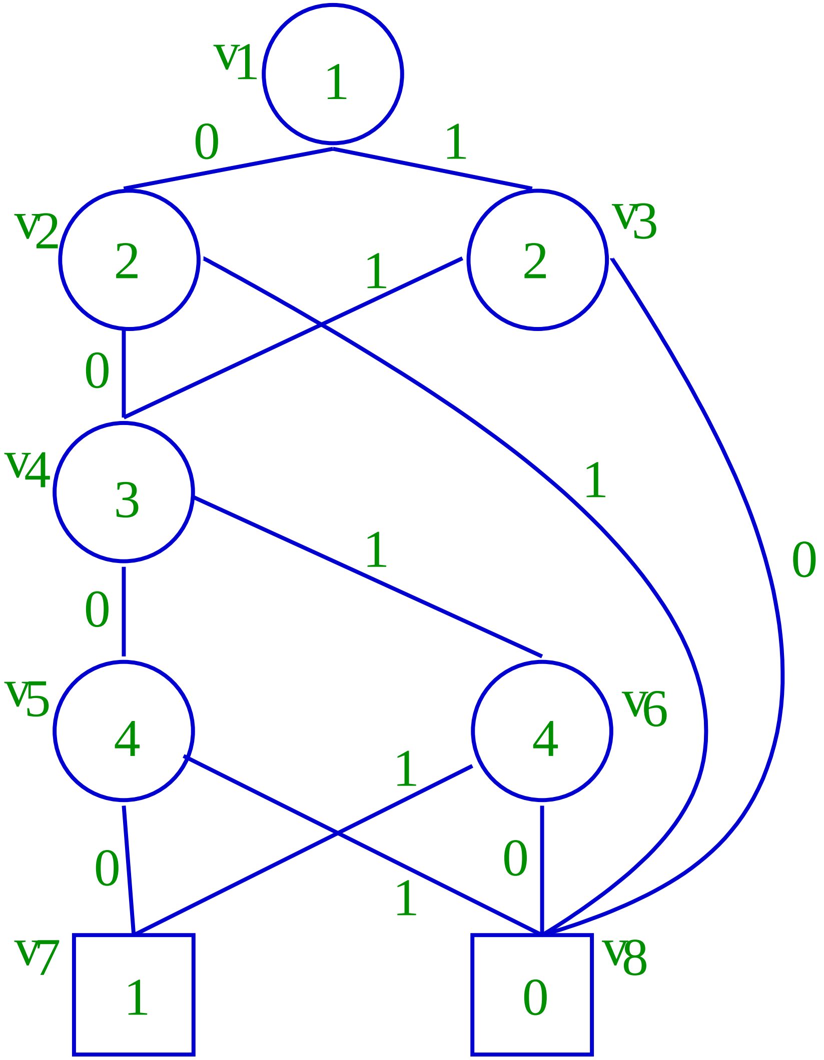
$$f_v(x_1, \ldots, x_n) = \bar{x}_i \cdot f_{low(v)}(x_1, \ldots, x_n) + x_i \cdot f_{high(v)}(x_1, \ldots, x_n).$$

# Correspondence (cont.)

- A path in the graph starting from the root is defined by a set of argument values.
- The value of the function for these arguments equals the value of the terminal vertex at the end of the path.
- Every vertex in the graph is contained in at least one path.

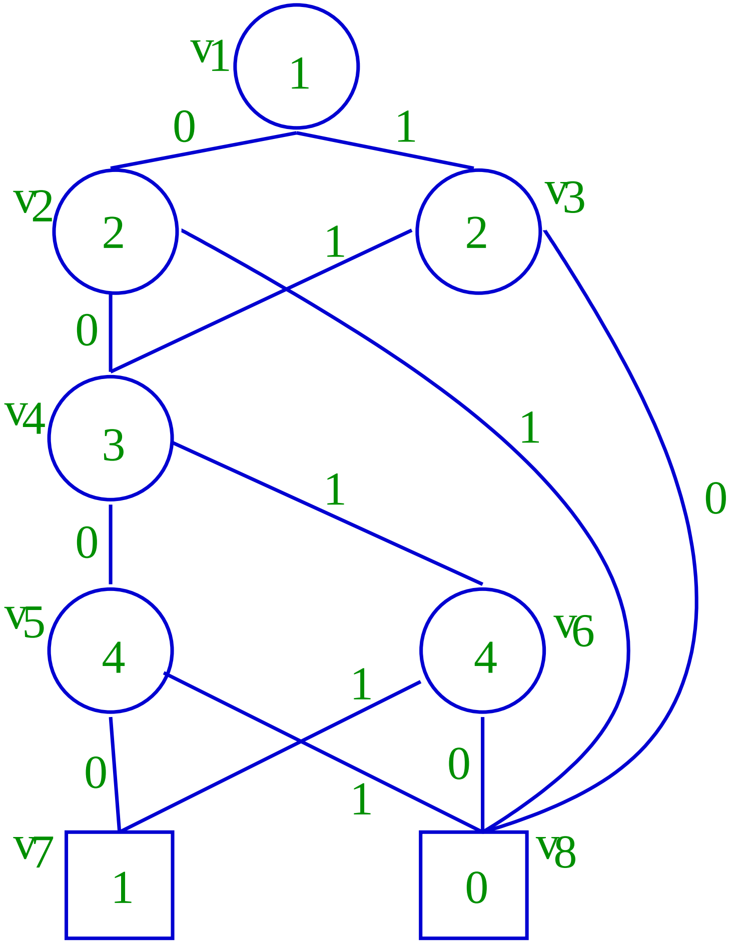
# Correspondence (cont.)

$$\begin{aligned}
f_{v_8} &= 0 \\
f_{v_7} &= 1 \\
f_{v_6} &= \bar{x}_4 \cdot f_{v_8} + x_4 \cdot f_{v_7} \\
&= x_4 \\
f_{v_5} &= \bar{x}_4 \cdot f_{v_7} + x_4 \cdot f_{v_8} \\
&= \bar{x}_4 \\
f_{v_4} &= \bar{x}_3 \cdot f_{v_5} + x_3 \cdot f_{v_6} \\
&= \bar{x}_3 \cdot \bar{x}_4 + x_3 \cdot x_4 \\
&\ldots \\
&\ldots \\
&\ldots \\
f_{v_1} &= (\bar{x}_1 \cdot \bar{x}_2 + x_1 \cdot x_2) \cdot (\bar{x}_3 \cdot \bar{x}_4 + x_3 \cdot x_4)
\end{aligned}$$



# Subgraph

- For any vertex $v$ in a function graph $G$, the subgraph rooted at $v$, denoted by $sub(G, v)$ is defined as the graph consisting of $v$ and all its descendants.



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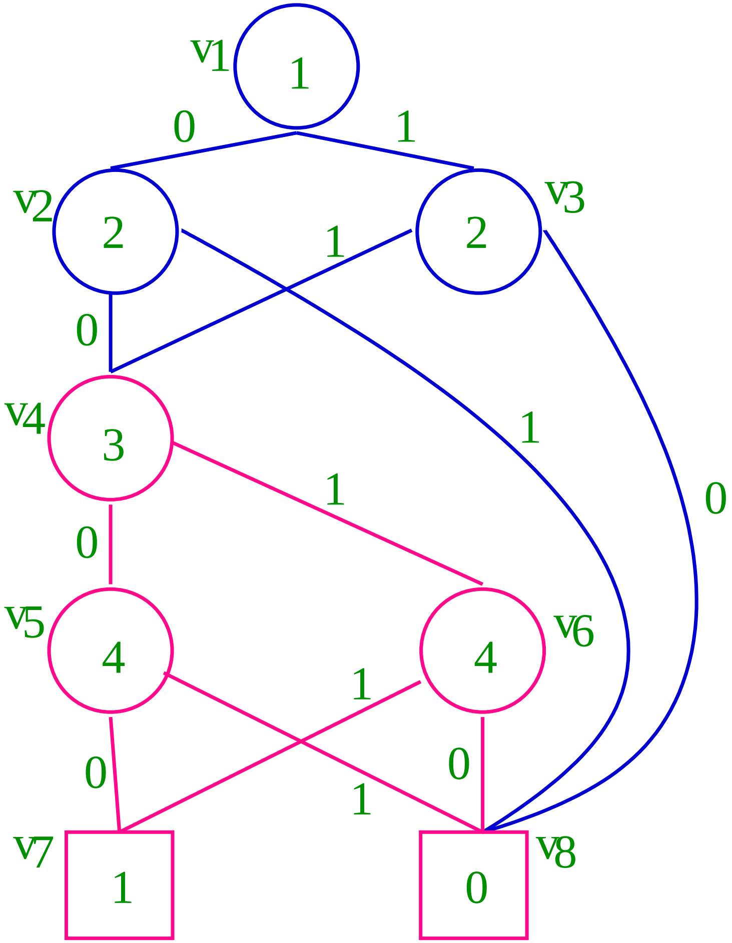
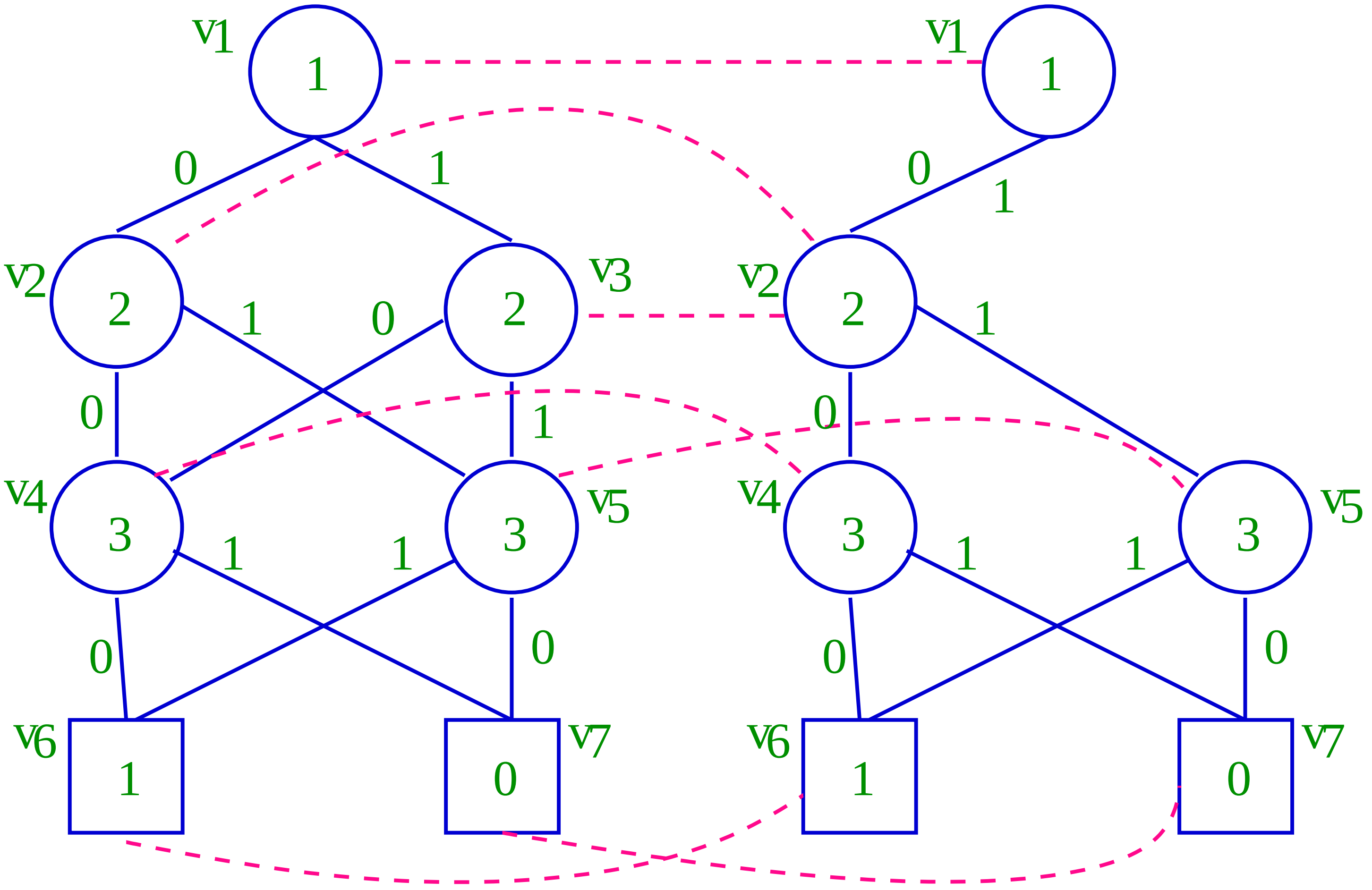


# Isomorphism

- Function graphs $G$ and $G'$ are isomorphic, denoted by $G \sim G'$, if there exists a one-to-one function $\sigma$ from vertices of $G$ onto the vertices of $G'$ such that for any vertex $v$ if $\sigma(v) = v'$, then either
  - both $v$ and $v'$ are terminal vertices with $value(v) = value(v')$, or
  - both $v$ and $v'$ are nonterminal vertices with $index(v) = index(v')$, $\sigma(low(v)) = low(v')$, and $\sigma(high(v)) = high(v')$

# Isomorphism (cont.)



Is this an isomorphic mapping? (parts of it are)

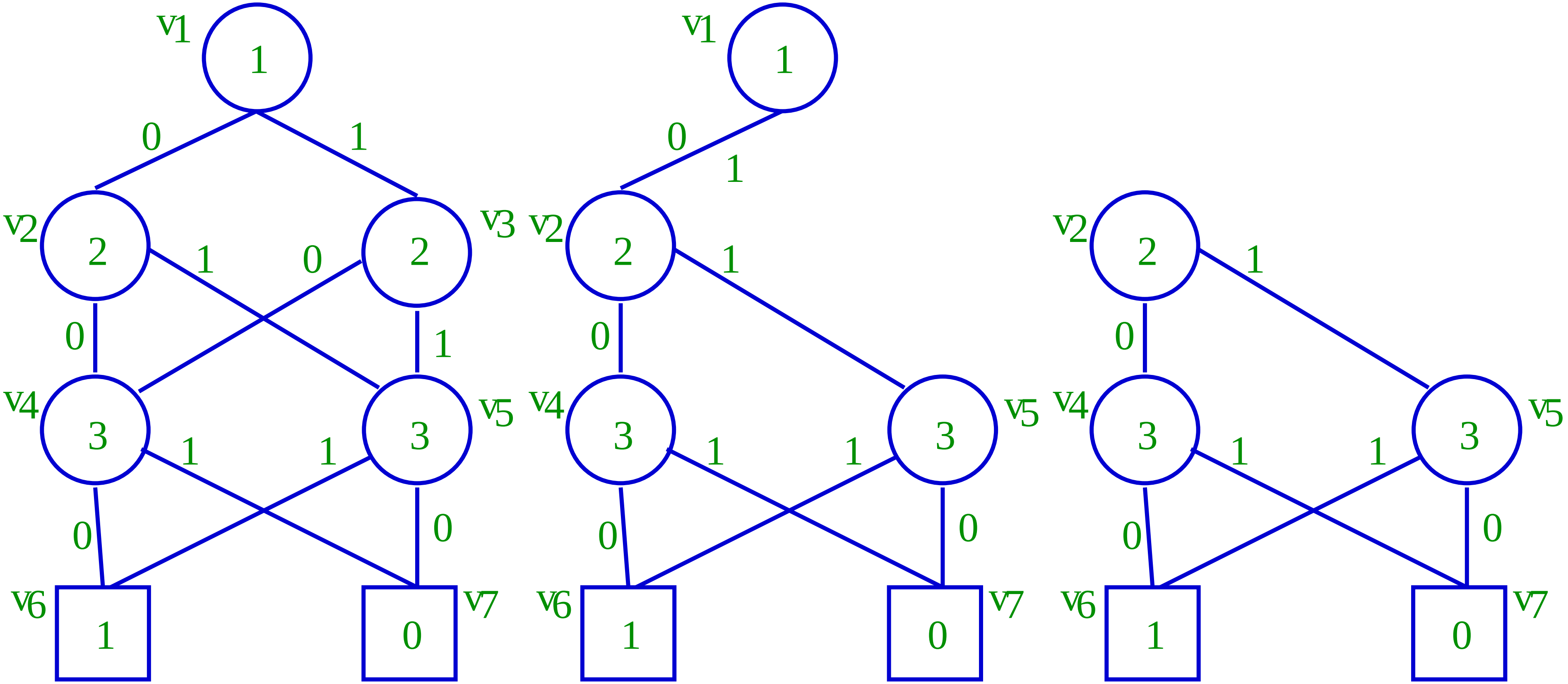
# Isomorphism (cont.)

- The isomorphic mapping $\sigma$ is quite constrained:
  - $r(G)$ must map to the $r(G')$,
  - $low(r(G))$ must map to $low(r(G'))$,
  - and so on all the way down to the terminal vertices.
- Lemma 1: If $G$ is isomorphic to $G'$ by mapping $\sigma$, denoted by $G \sim_\sigma G'$, then for any vertex $v$ in $G$, $sub(G, v) \sim sub(G', \sigma(v))$.

# Reduced Function Graph

- A function graph $G$ is reduced if
  - it contains no vertex $v$ with $low(v) = high(v)$,
  - nor does it contain distinct vertices $v$ and $v'$ such that the subgraphs rooted by $v$ and $v'$ are isomorphic.
- A reduced function graph is now commonly called (Reduced) OBDD.
- Lemma 2: For every vertex $v$ in a reduced function graph $G$, $sub(G, v)$ is itself a reduced function graph.

# Reduced Function Graph (cont.)



# Canonical Form

- Theorem: For any Boolean function $f$, there is a unique (up to isomorphism) reduced function graph denoting $f$ and any other function graph denoting $f$ contains more vertices.

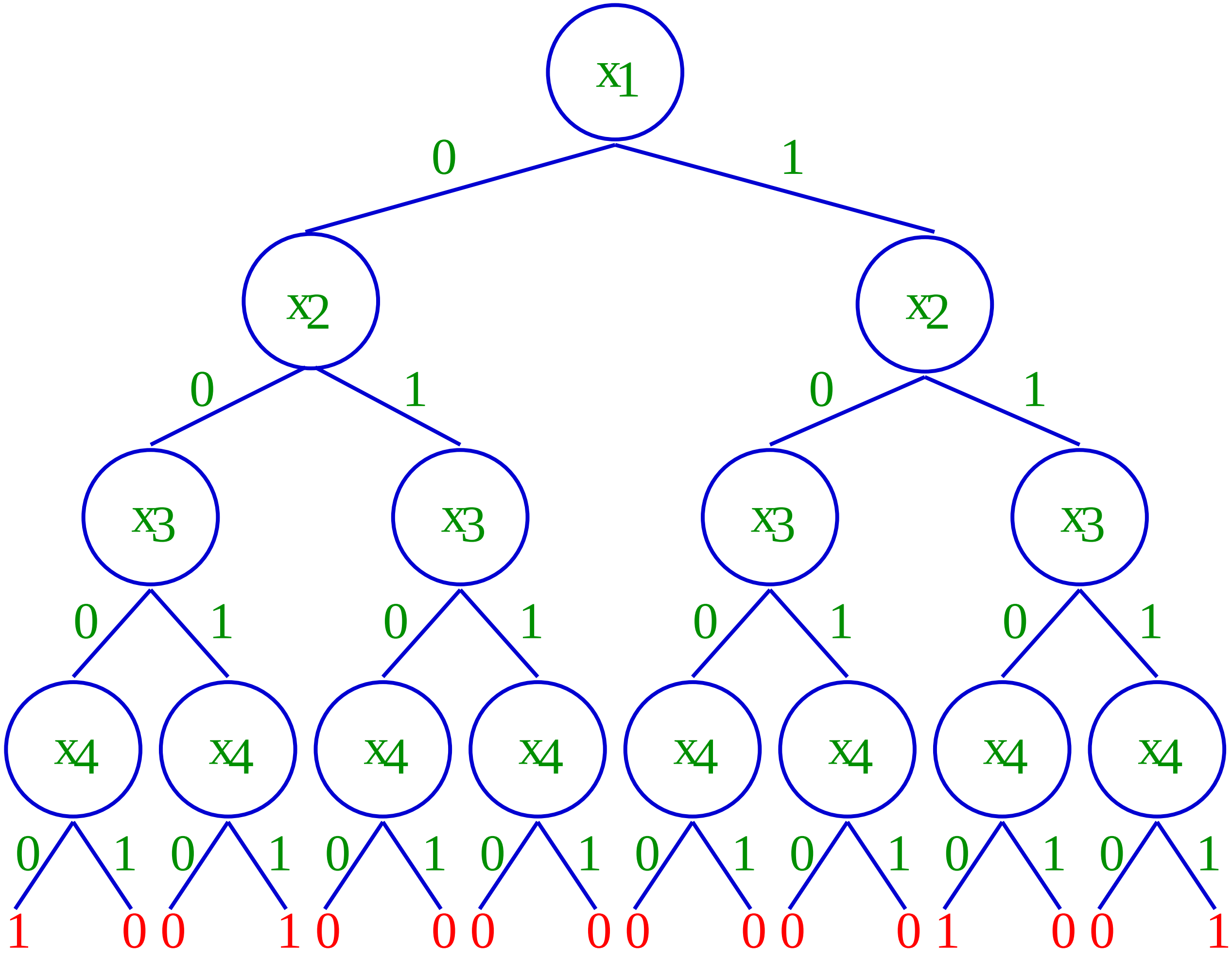
# Basic Operations

| **Procedure** | **Result** | **Time Complexity** |
|---|---|---|
| Reduce | $G$ reduced to canonical form | $O(\|G\| \cdot \log \|G\|)$ |
| Apply | $f_1 \langle op \rangle f_2$ | $O(\|G_1\| \cdot \|G_2\|)$ |
| Restrict | $f\,\|_{x_i = b}$ | $O(\|G\| \cdot \log \|G\|)$ |
| Compose | $f_1\,\|_{x_i = f_2}$ | $O(\|G_1\|^2 \cdot \|G_2\|)$ |
| Satisfy-one | some element of $S_f$ | $O(n)$ |
| Satisfy-all | $S_f$ | $O(n \cdot \|S_f\|)$ |
| Satisfy-count | $\|S_f\|$ | $O(\|G\|)$ |

# Reduction

- The reduction algorithm transforms an arbitrary function graph into a reduced graph denoting the same function.
- The algorithm works from the terminal vertices up to the root:
    - Remove duplicate terminals (terminal vertices $v$ and $u$ such that $value(v) = value(u)$).
    - Remove duplicate nonterminals (nonterminal vertices $v$ and $u$ such that $index(v) = index(u)$, $id(low(v)) = id(low(u))$, and $id(high(v)) = id(high(u))$).
    - Remove duplicate tests (a nonterminal vertex $v$ such that $low(v) = high(v)$).

# A Reduction Example



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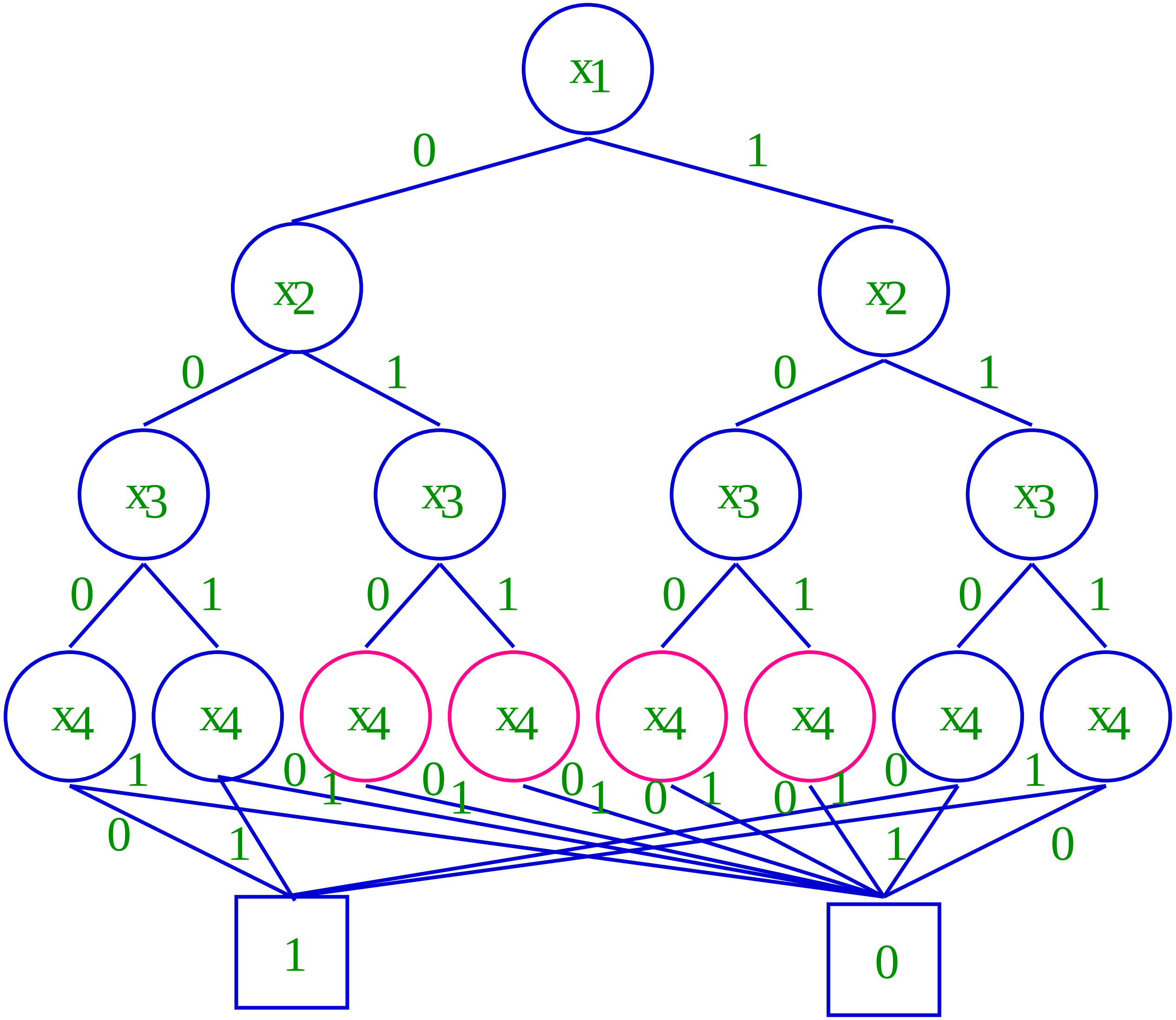

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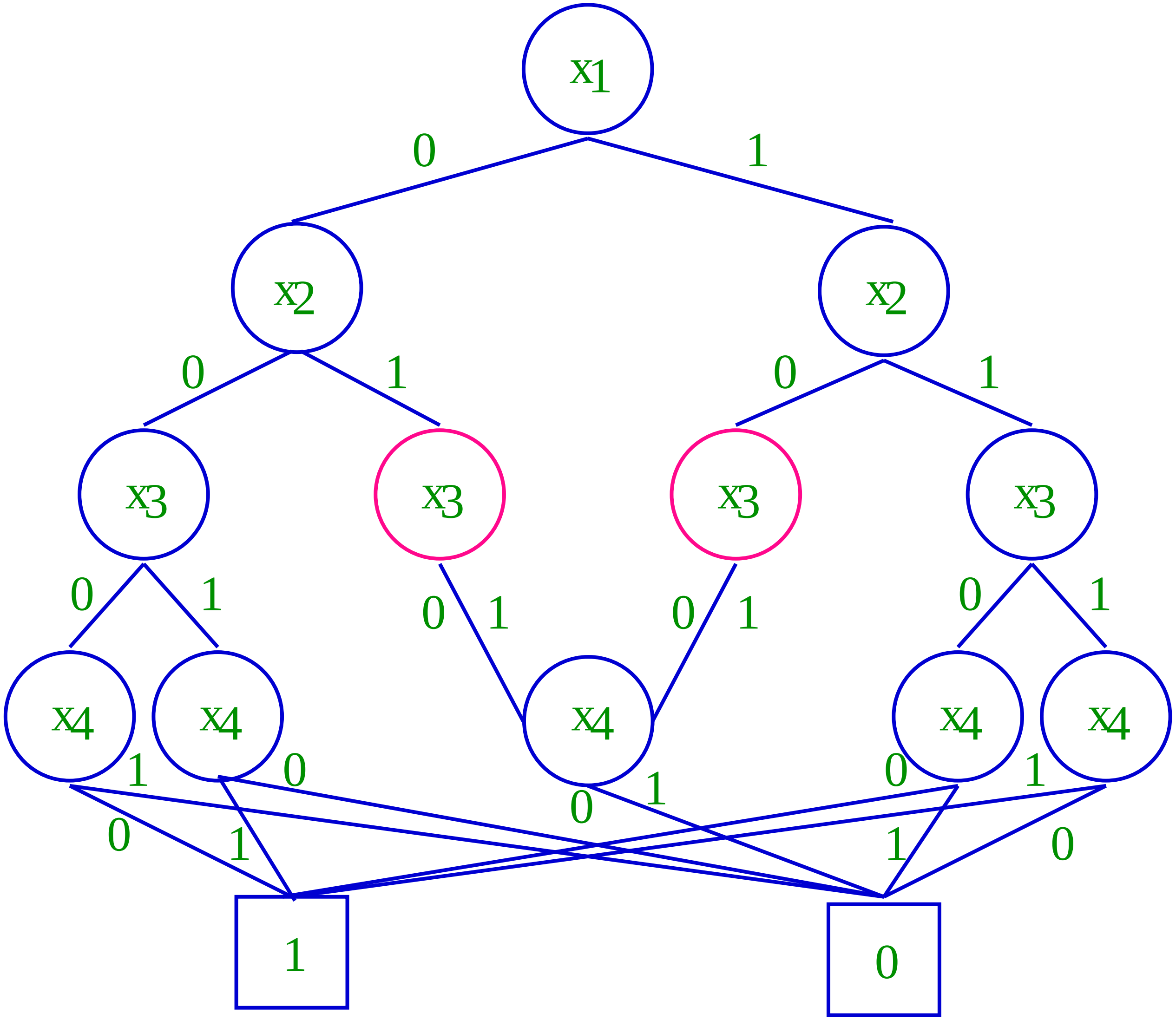
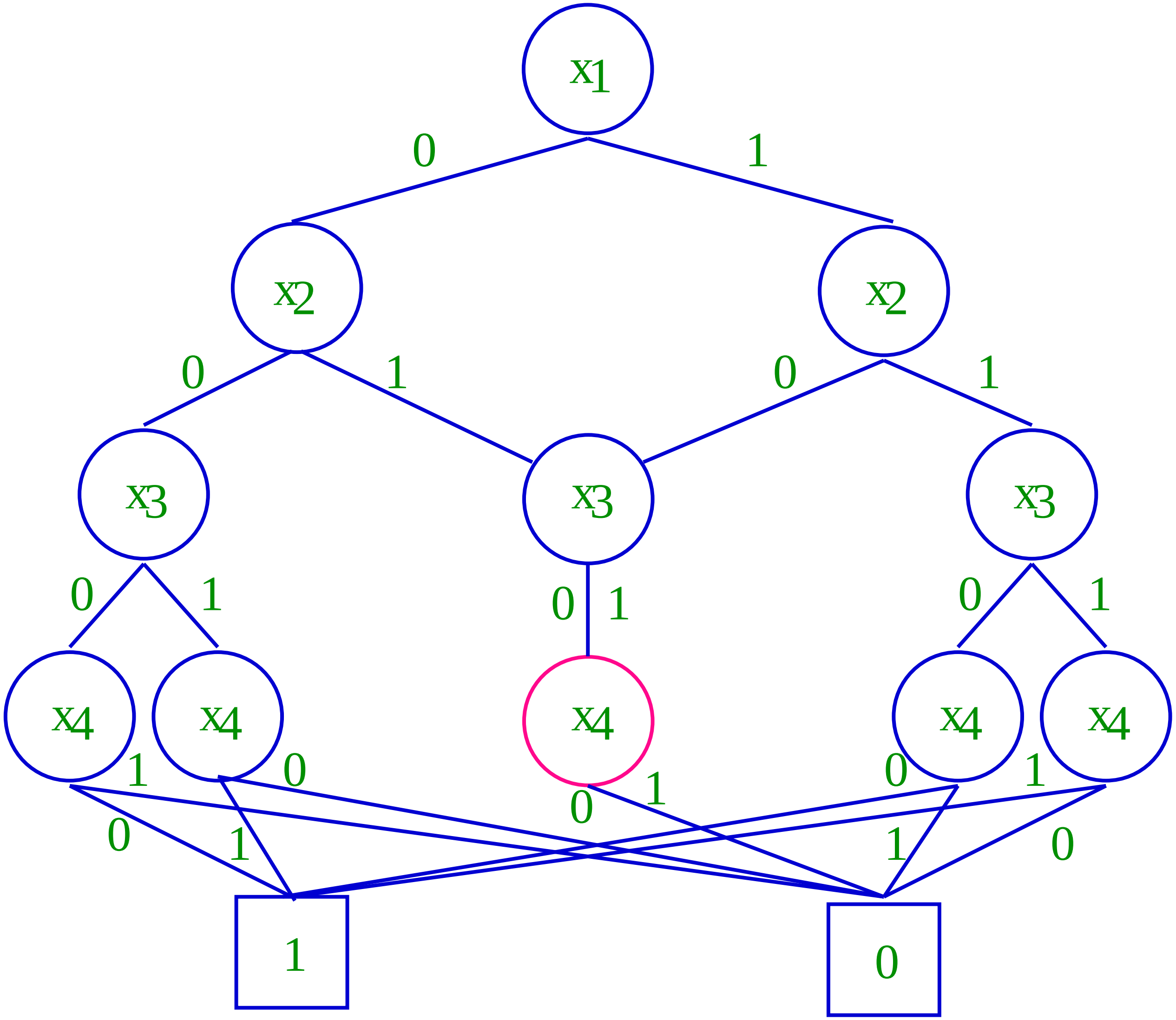


Note: not strictly bottom to top (for better layouts).

# A Reduction Example



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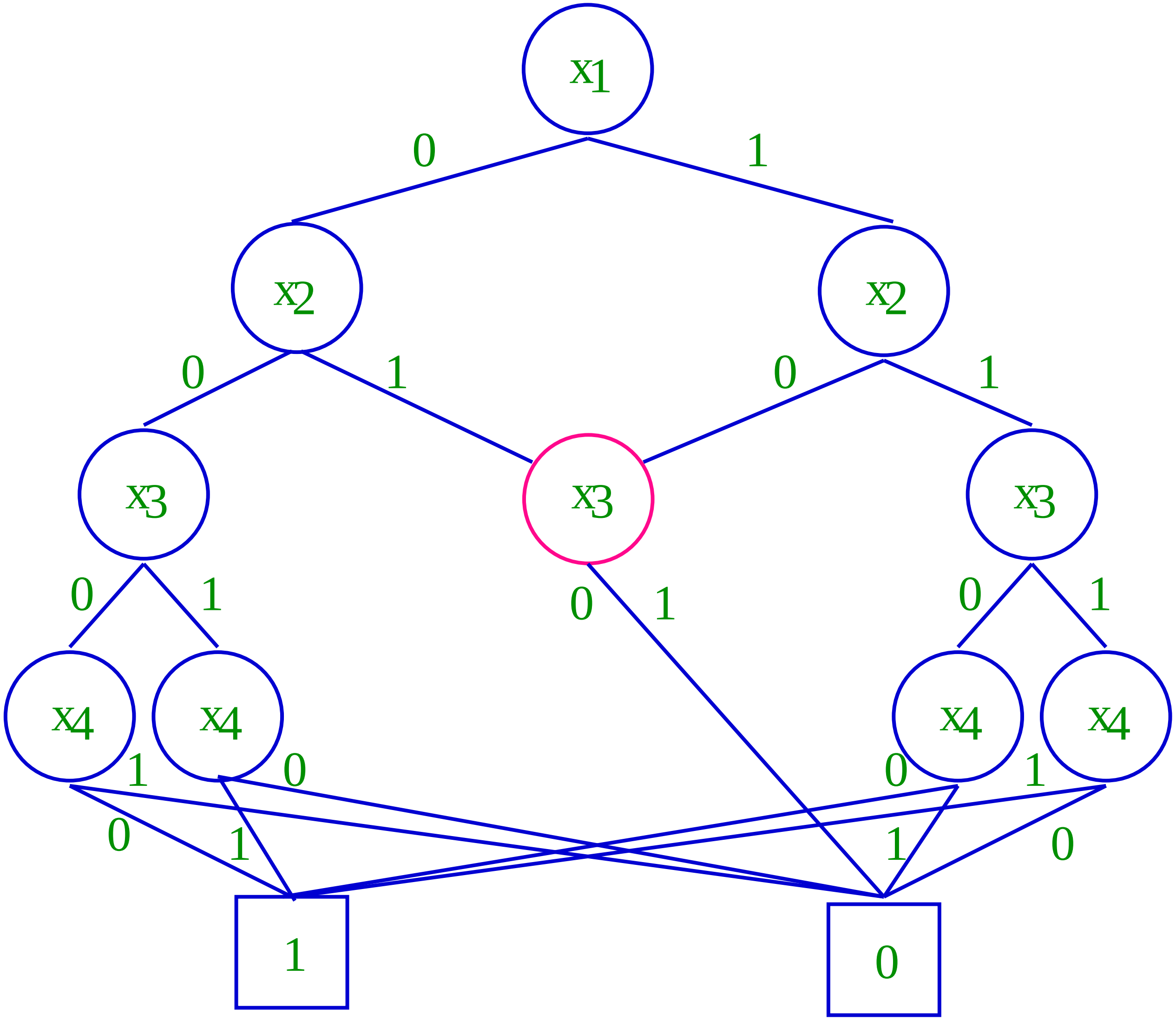

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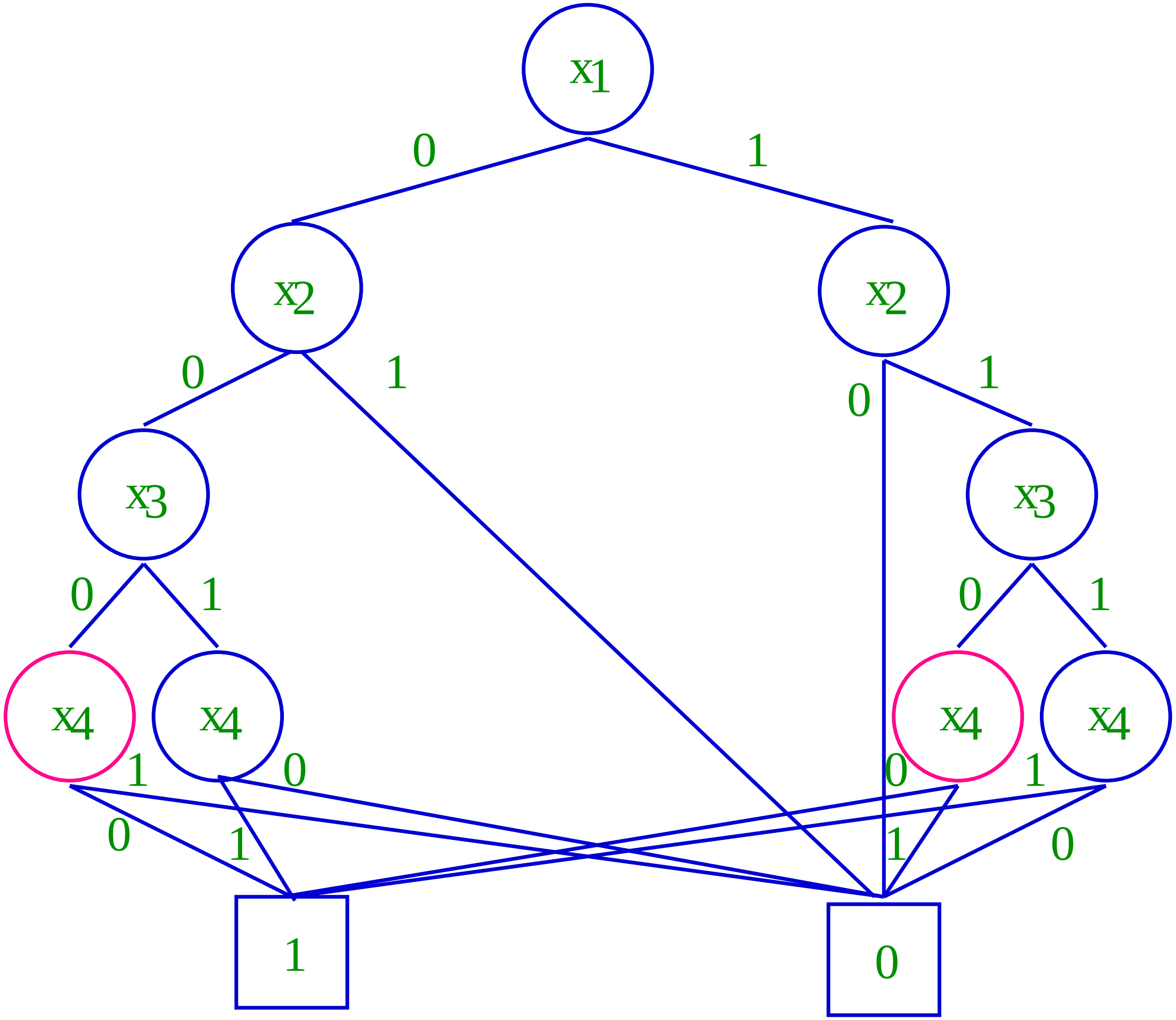

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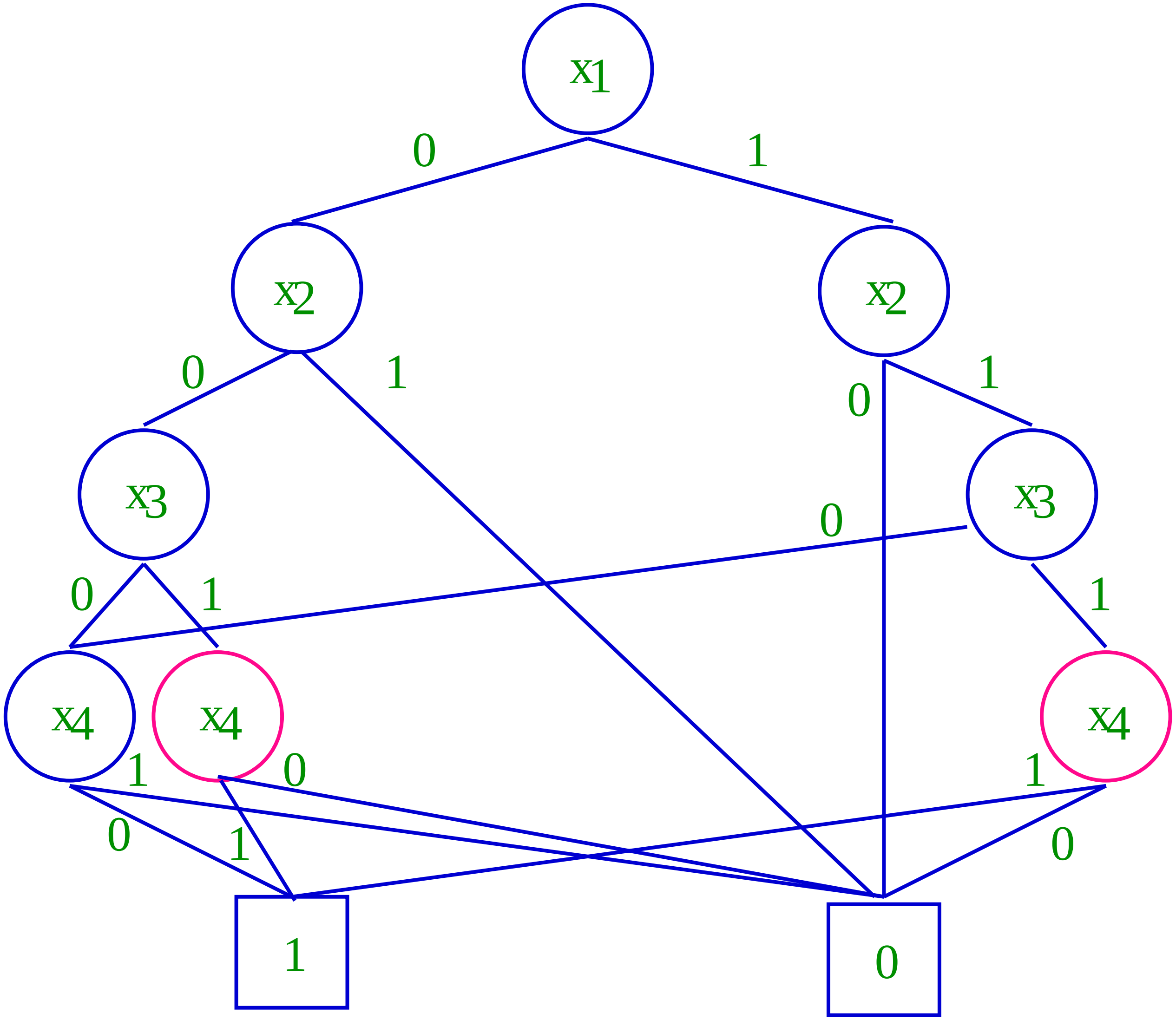

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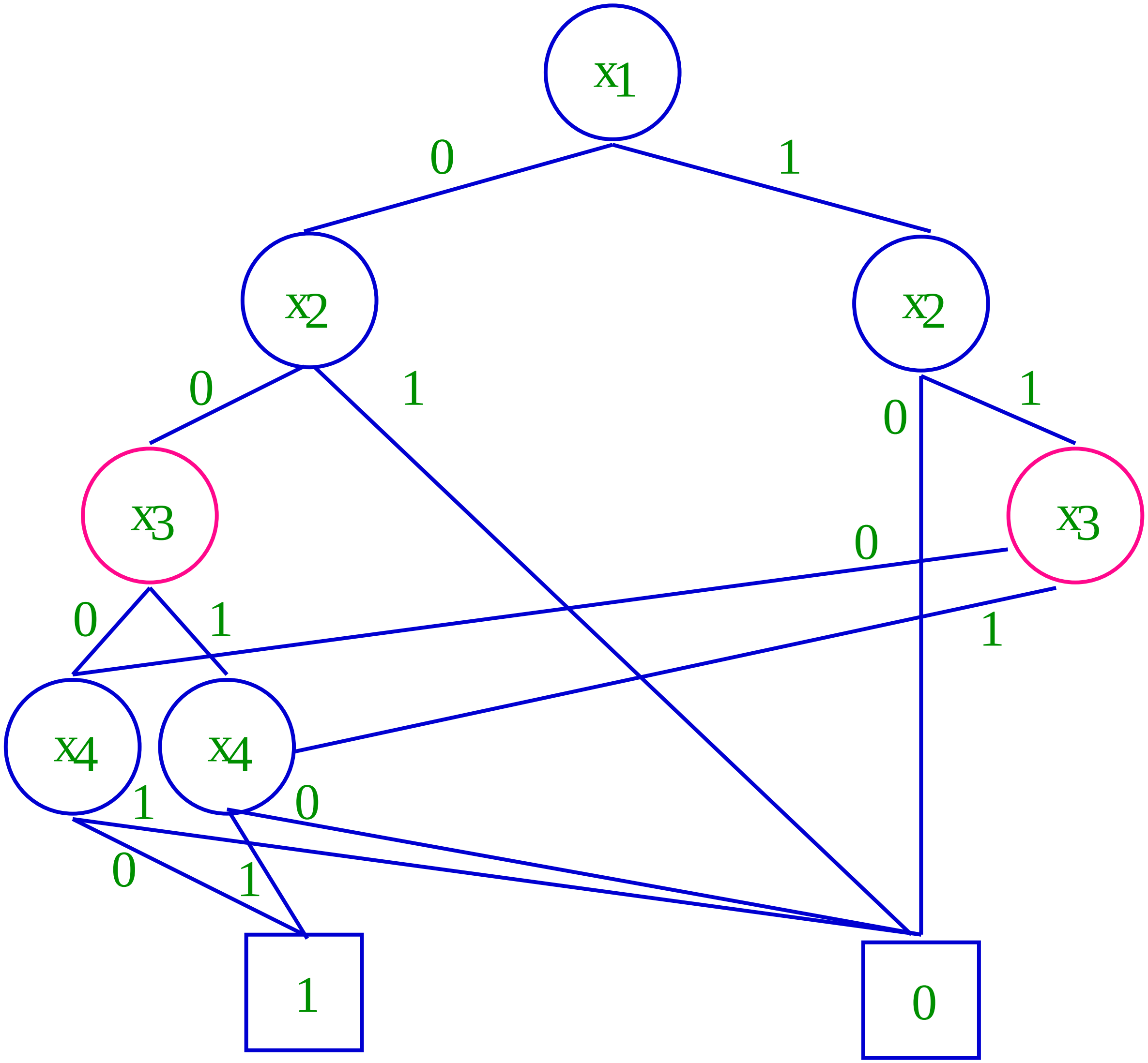

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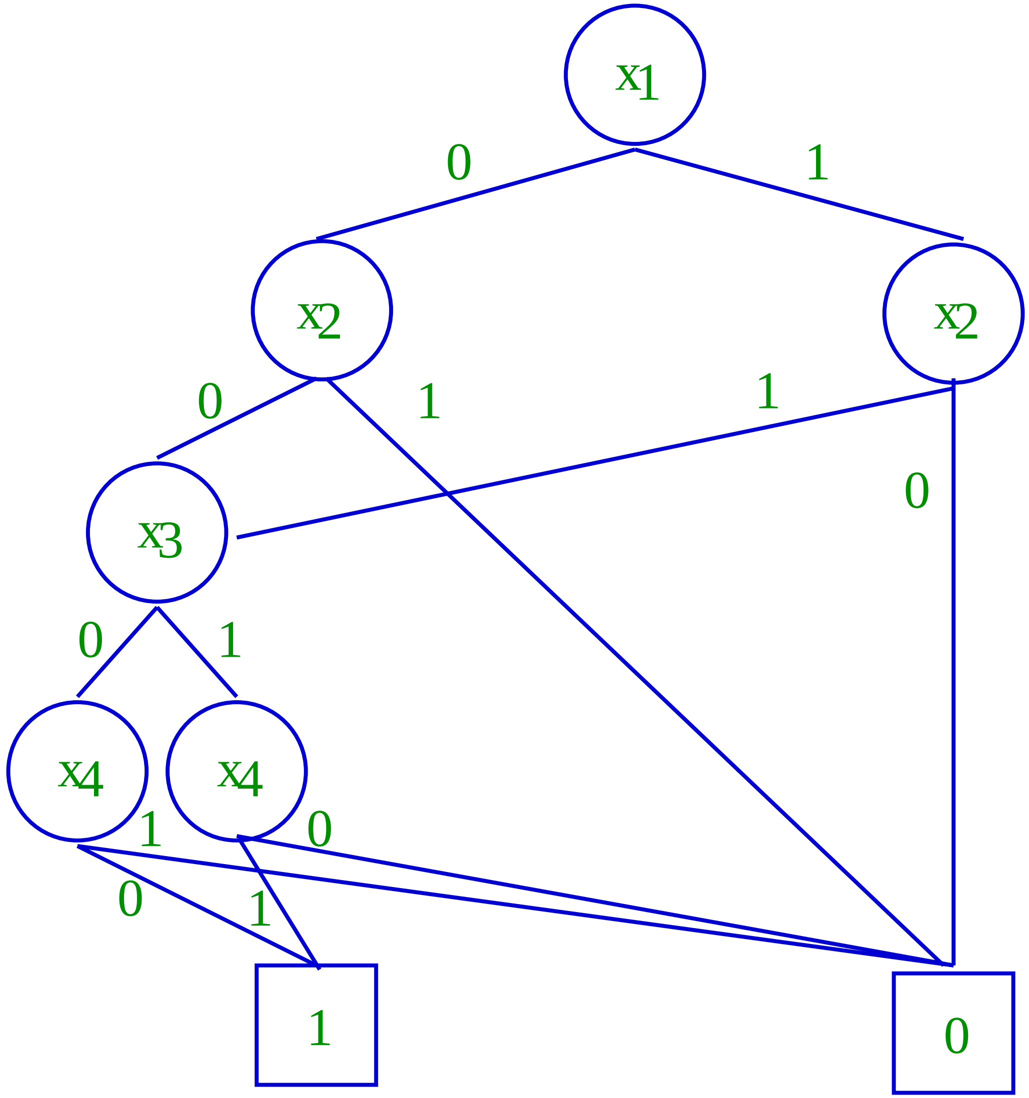

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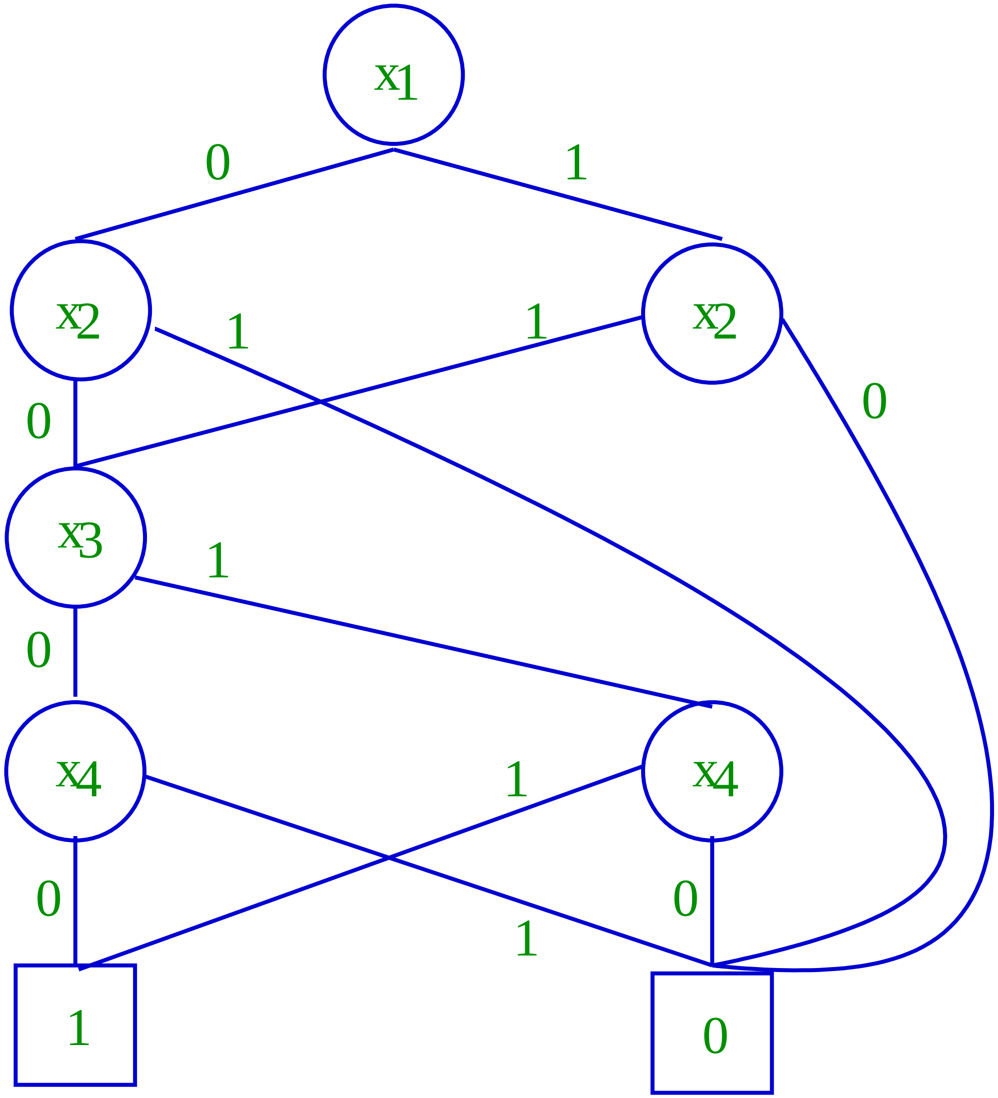


# Apply

- The procedure *Apply* takes graphs representing functions $f_1$ and $f_2$, a binary operator $\langle op \rangle$ and produces a reduced graph representing the function $f_1 \langle op \rangle f_2$ defined as:

$$[f_1 \langle op \rangle f_2](x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) \langle op \rangle f_2(x_1, \ldots, x_n).$$

- Derive a recursive expansion from the Shannon expansion:

$$f_1 \langle op \rangle f_2 = \bar{x}_i \cdot (f_1|_{x_i=0} \langle op \rangle f_2|_{x_i=0}) + x_i \cdot (f_1|_{x_i=1} \langle op \rangle f_2|_{x_i=1})$$

# Apply (cont.)

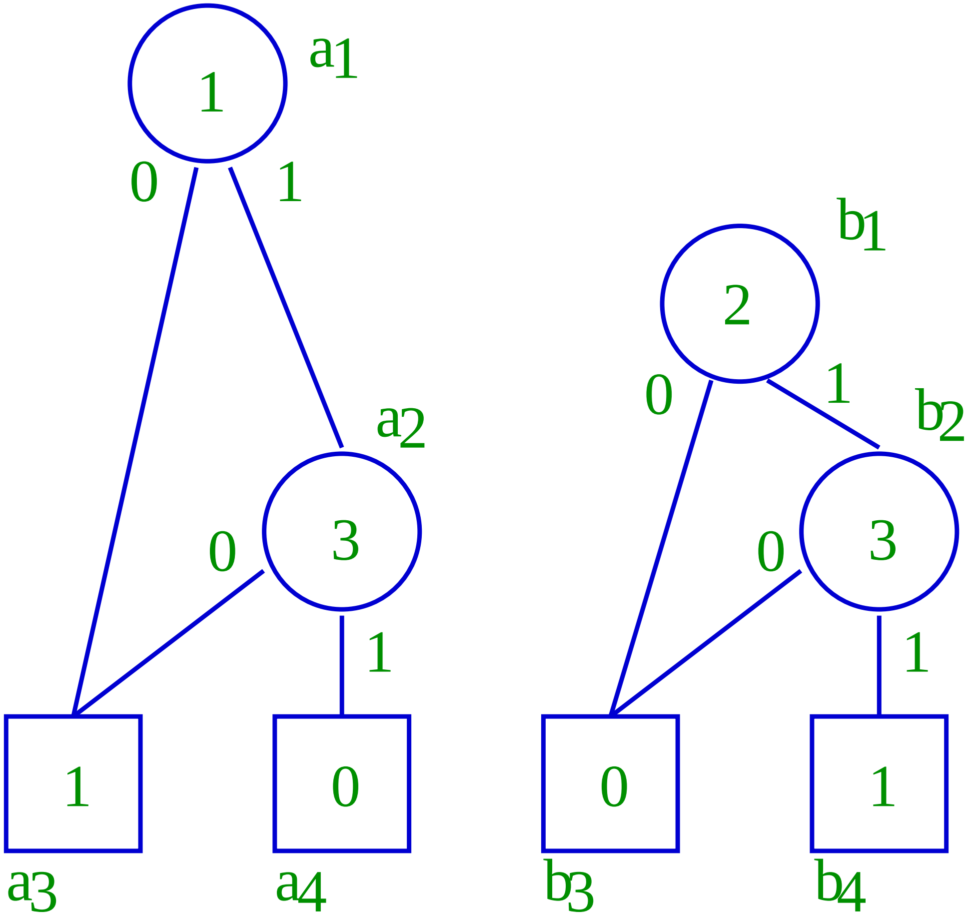
```
function Apply($v1$, $v2$: vertex $\langle op \rangle$: operator): vertex
{var $T$: array[$1..|G_1|$, $1..|G_2|$] of vertex;}
begin
        Initialize all elements of $T$ to null;
        $u$ := Apply-step($v1$, $v2$);
        return$(Reduce(u))$;
end;
```

# Apply (cont.)

```
function Apply-step($v1$, $v2$: vertex): vertex;
begin
    $u$:= $T[v1.id$, $v2.id]$;
    if $u \neq null$ then return$(u)$; {have already evaluated}
    $u$:= new vertex record; $u.mark$ := false;
    $T[v1.id$, $v2.id]$ := $u$; {add vertex to table}
    $u.value$ := $v1.value \langle op \rangle v2.value$;
    if $u.value \neq X$
        then $u.index$ := $n + 1$; $u.low$ := $null$; $u.high$ := $null$;
    else {create nonterminal and evaluate further down}
        $u.index$ := Min($v1.index$, $v2.index$);
        if $v1.index$ = $u.index$
            then begin $vlow1$ := $v1.low$; $vhigh1$ := $v1.high$ end
            else begin $vlow1$ := $v1$; $vhigh1$ := $v1$ end;
        if $v2.index$ = $u.index$
            then begin $vlow2$ := $v2.low$; $vhigh2$ := $v2.high$ end
            else begin $vlow2$ := $v2$; $vhigh2$ := $v2$ end;
        $u.low$ := Apply-step($ulow1$, $vlow2$);
        $u.high$ := Apply-step($vhigh1$, $vhigh2$);
    return$(u)$;
end;
```

# An Apply Example

$$\overline{x_1 \cdot x_3} \quad + \quad x_2 \cdot x_3 \quad = \quad \overline{x_1 \cdot \overline{x_2} \cdot x_3}$$



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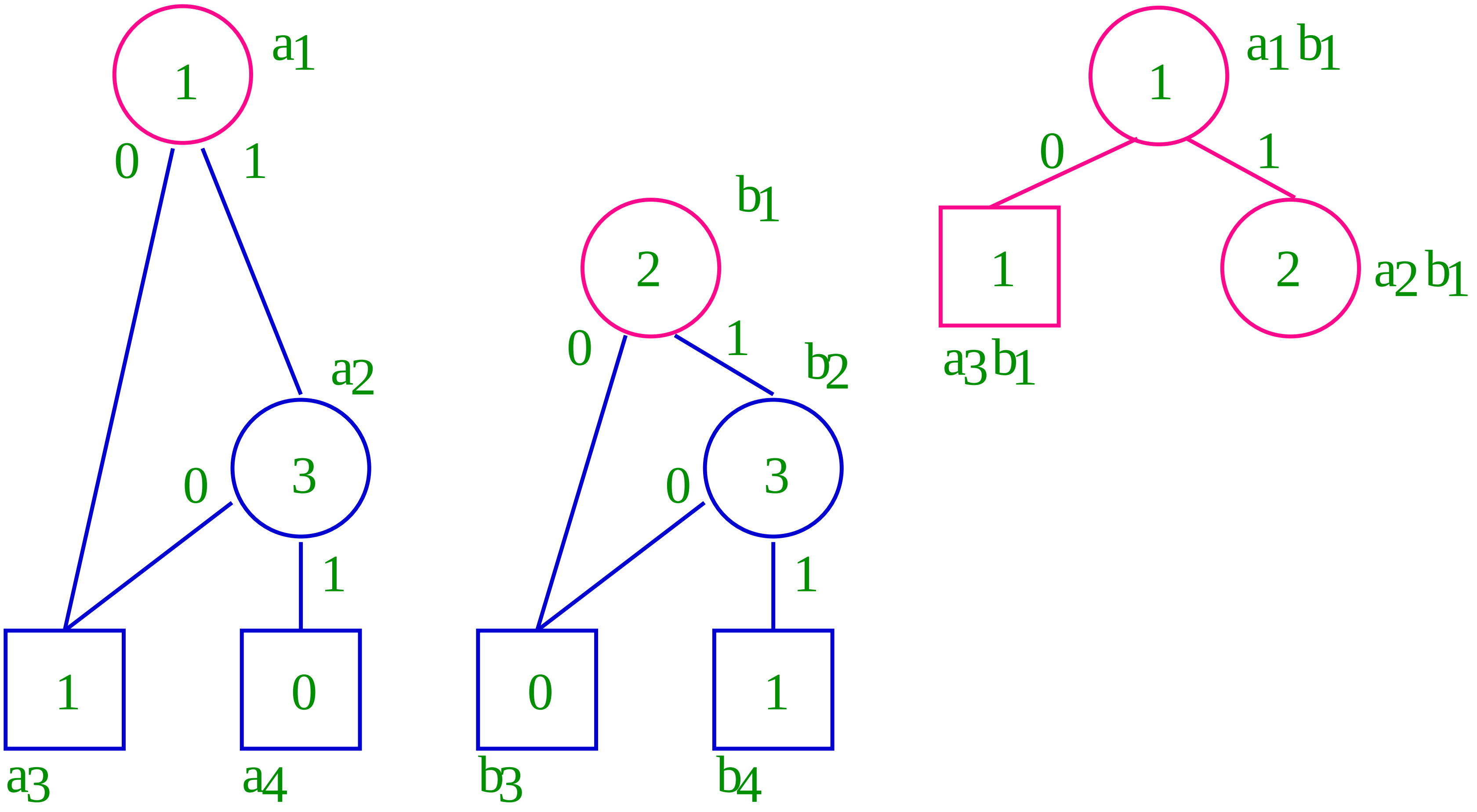
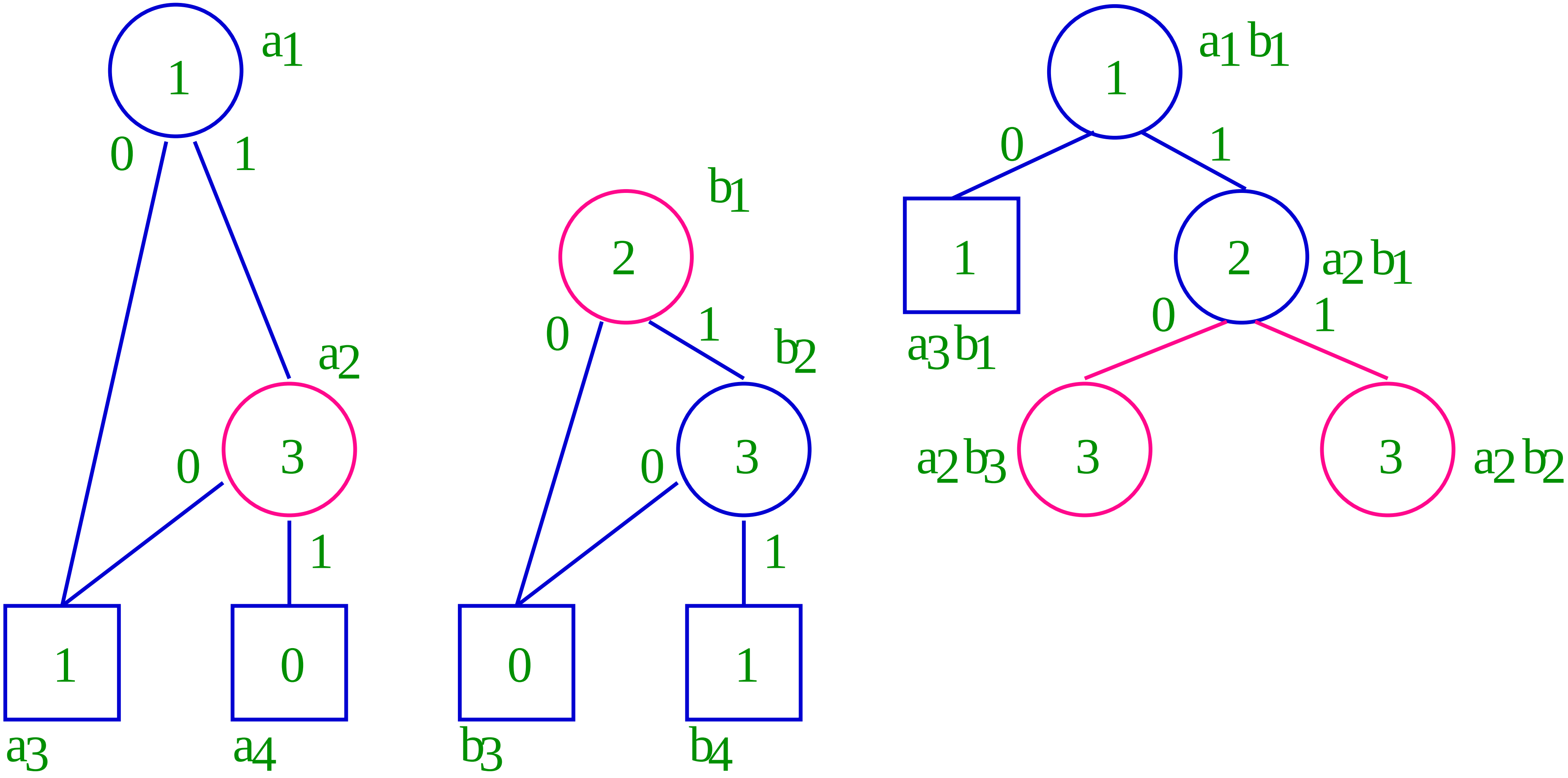


# An Apply Example

$\overline{x_1 \cdot x_3}$ + $x_2 \cdot x_3$ = $\overline{x_1 \cdot \overline{x_2} \cdot x_3}$



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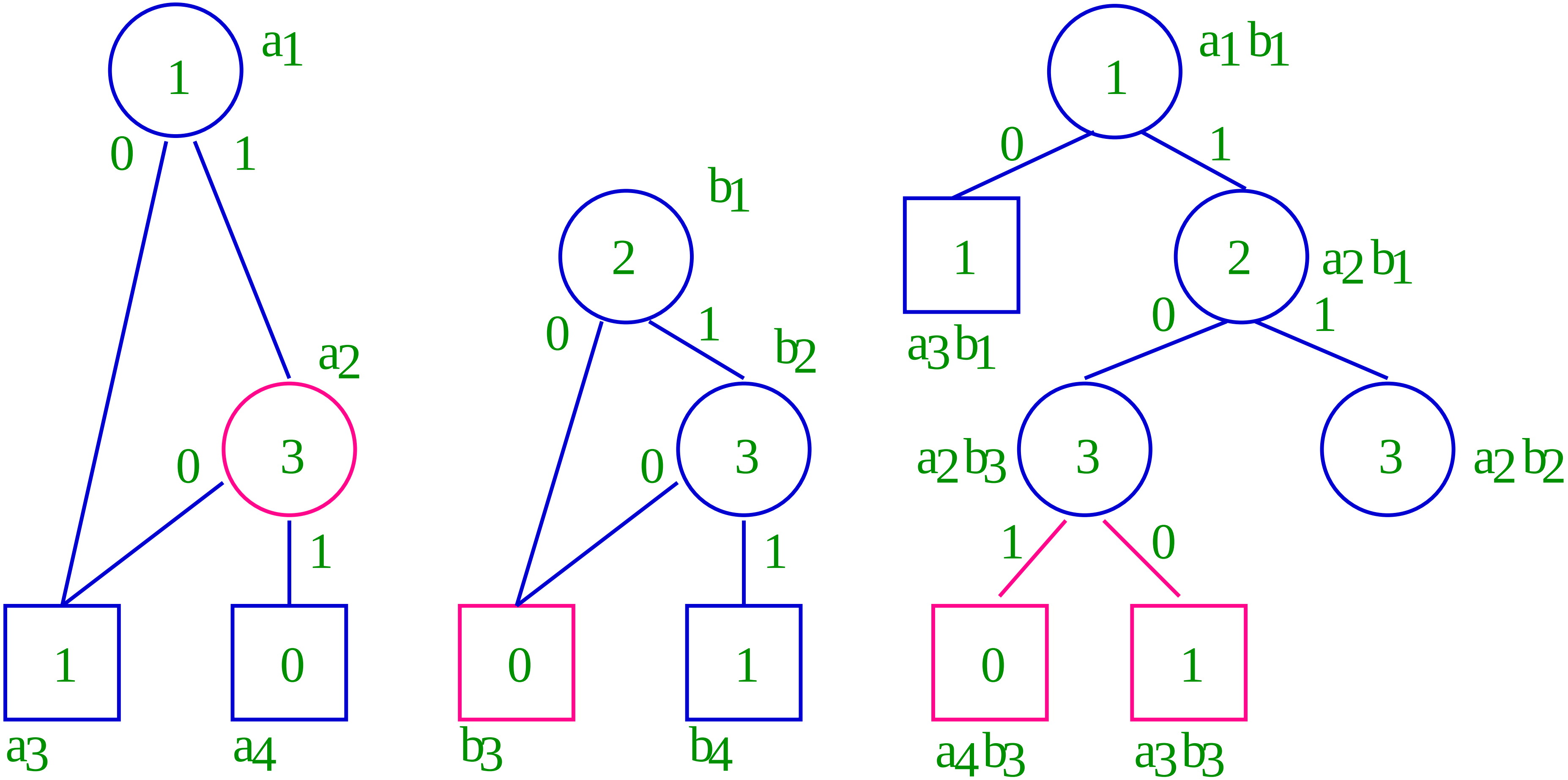

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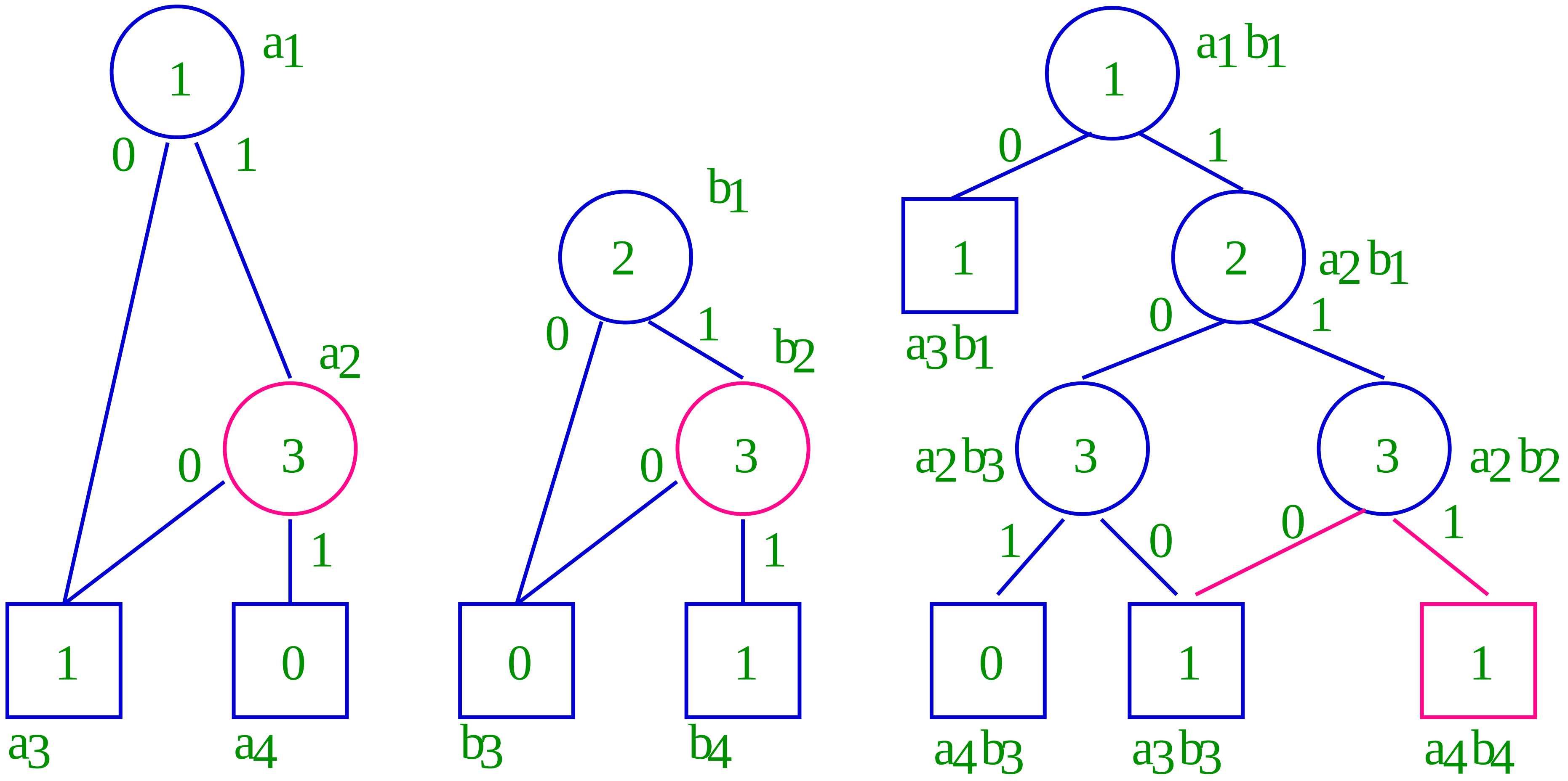

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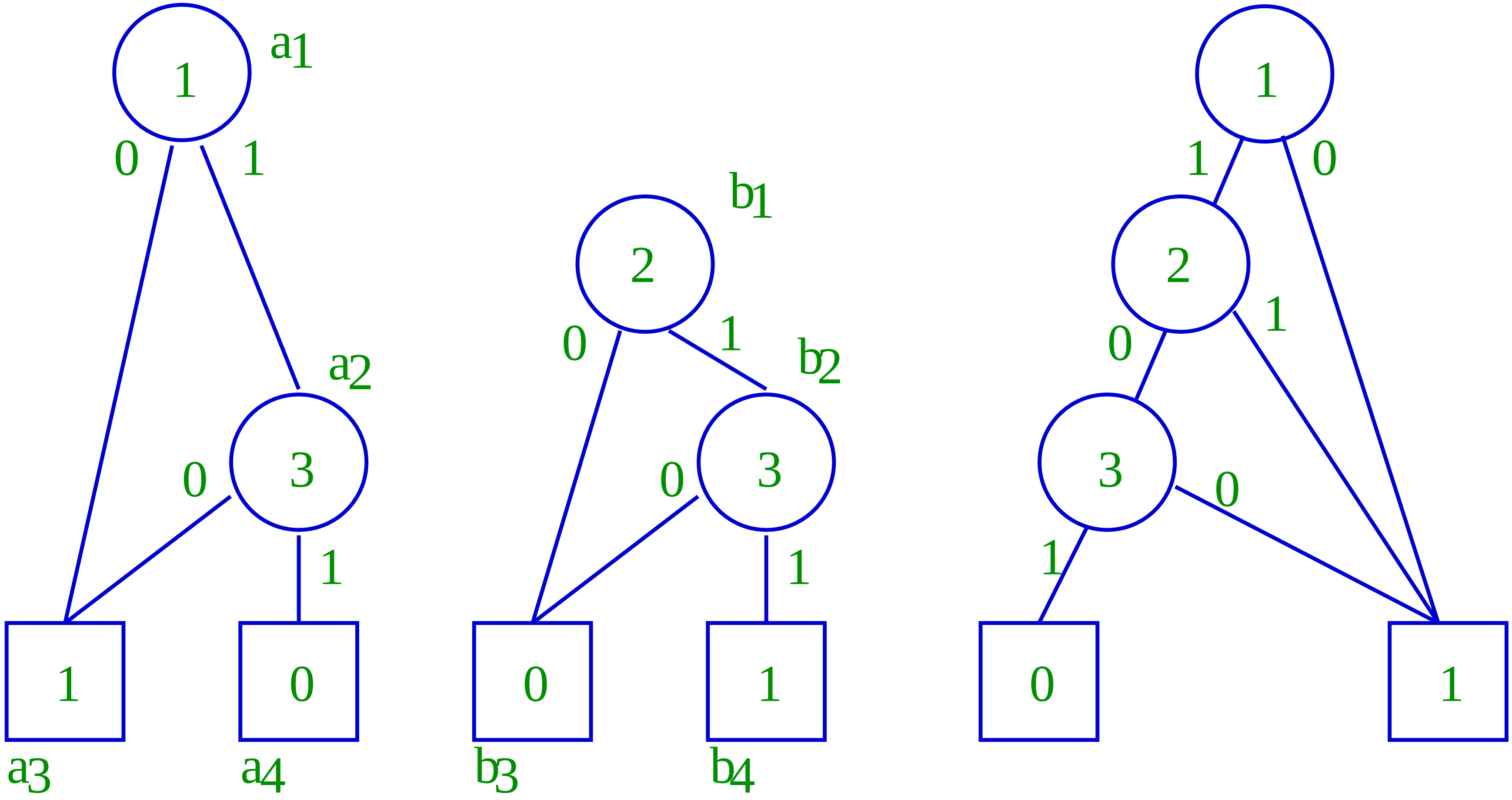
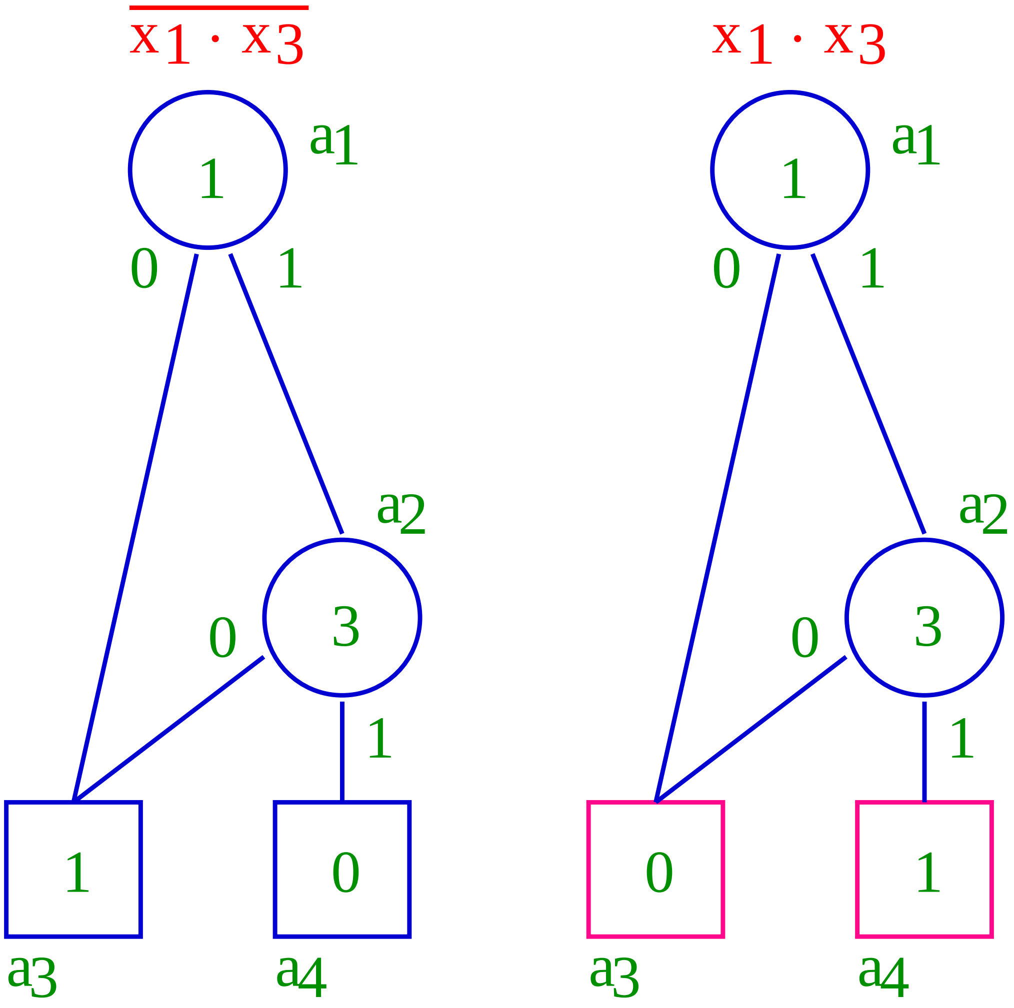


# Complementation

- To complement an OBDD, simply complement its terminal vertices.

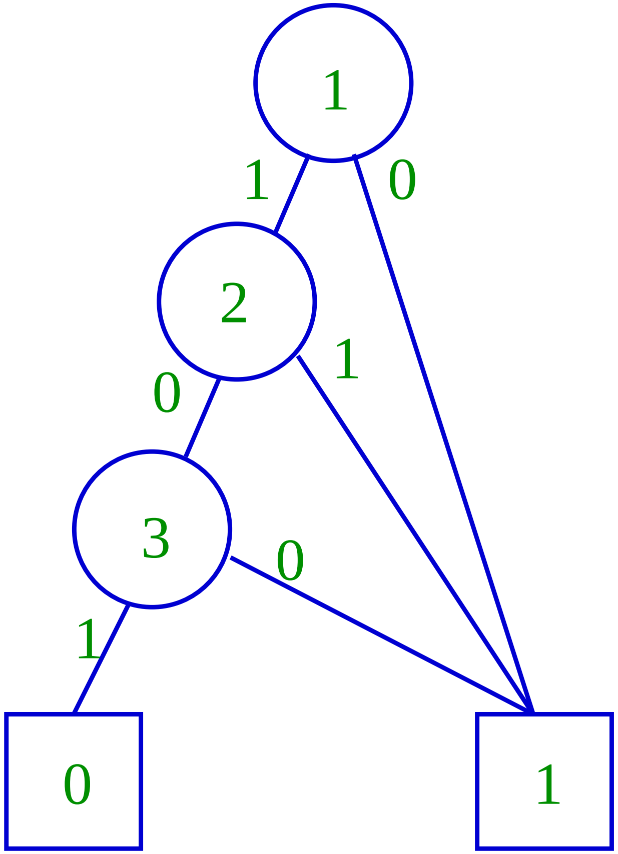


# Restriction

- The procedure *Restrict* transforms the graph representing a function $f$ into one representing the function $f|_{x_i=b}$.
- Steps of *Restrict*:
  - Look for a vertex $v$ with $index(v) = i$.
  - Change it to point either to $low(v)$ (for $b = 0$) or to $high(v)$ (for $b = 1$).
  - After changing every vertex $v$ with $index(v) = i$, run the reduction procedure.

# A Restriction Example

$$\left.\overline{x_1 \cdot \overline{x_2} \cdot x_3}\right|_{x_2 = 0} = \overline{x_1 \cdot x_3}$$



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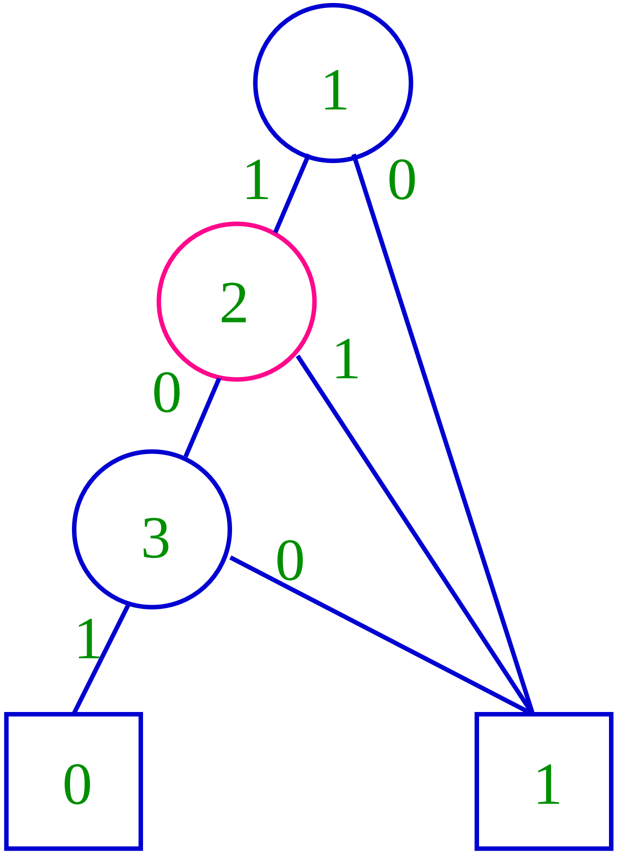

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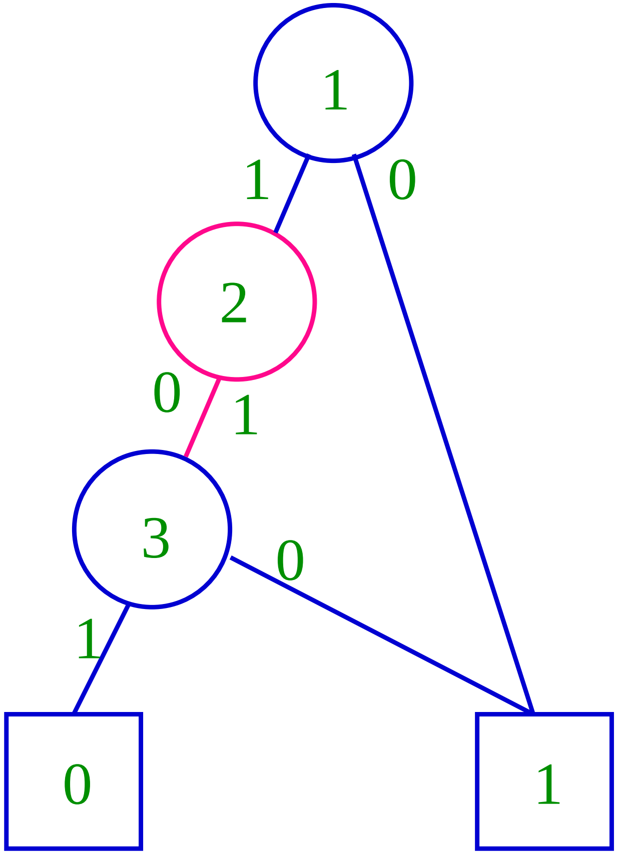

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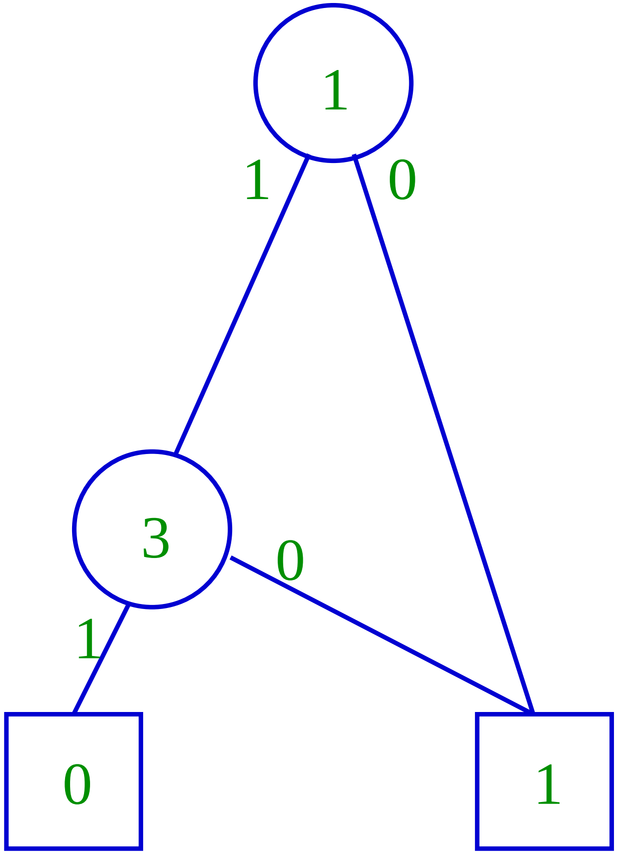


# Composition

- The procedure *Compose* constructs the graph for the function obtained by composing two functions.
- Composition can be expressed in terms of restriction and Boolean operations according to the following expansion:

$$f_1 |_{x_i = f_2} = f_2 \cdot f_1 |_{x_i = 1} + (\neg f_2) \cdot f_1 |_{x_i = 0}$$

- It is sufficient to use *Restrict* and *Apply* to implement *Compose*.

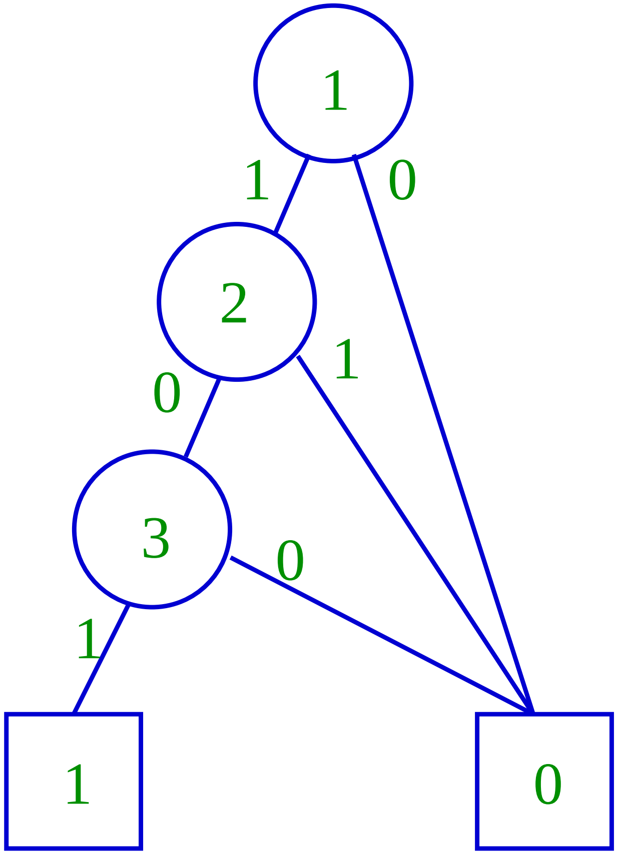
# Satisfy-one

- The *Satisfy-one* procedure utilizes a classic depth-first search with backtracking.

```
function Satisfy-one(v: vertex; x: array[1..n] of integer): boolean
begin
    if value(v) = 0 then return false;
    if value(v) = 1 then return true;
    x[i] := 0;
    if Satisfy-one(low(v), x) then return true;
    x[i] := 1;
    return Satisfy-one(high(v), x);
end;
```

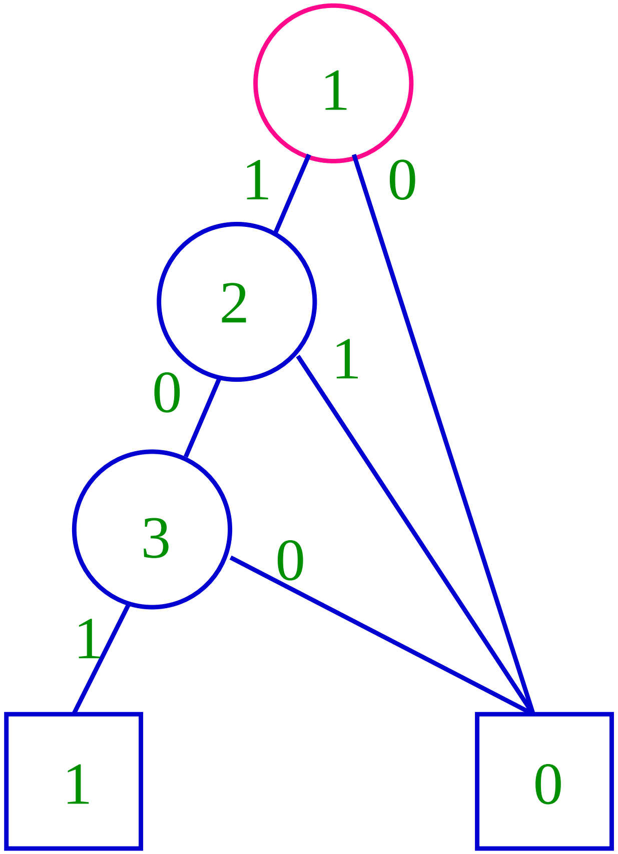
# A Satisfy-one Example

$x_1 \cdot \overline{x_2} \cdot x_3$



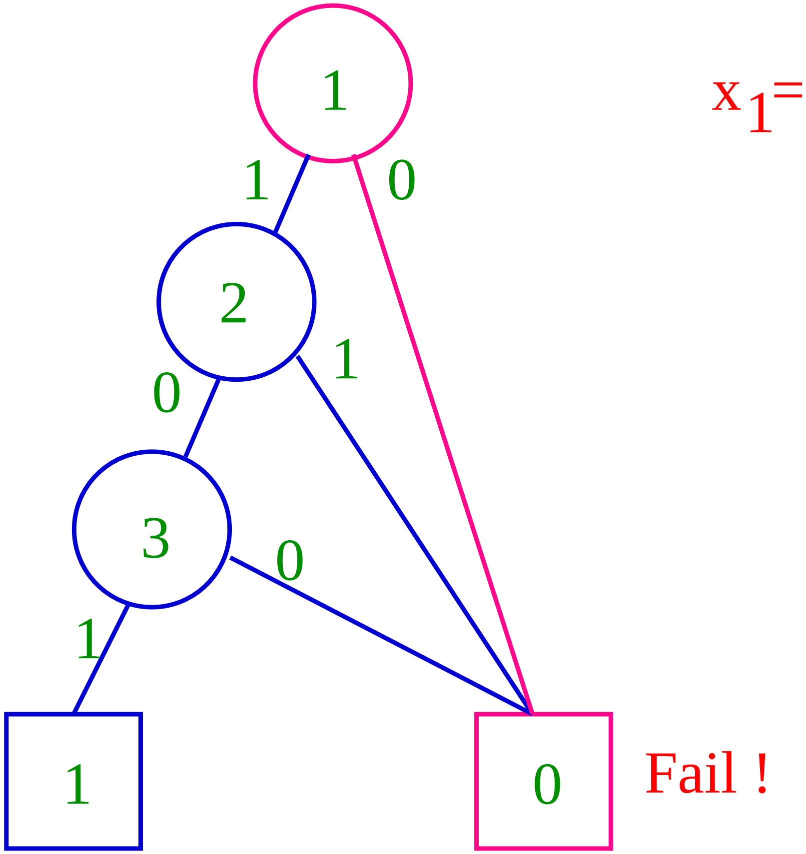
# A Satisfy-one Example

$x_1 \cdot \overline{x_2} \cdot x_3$



# A Satisfy-one Example

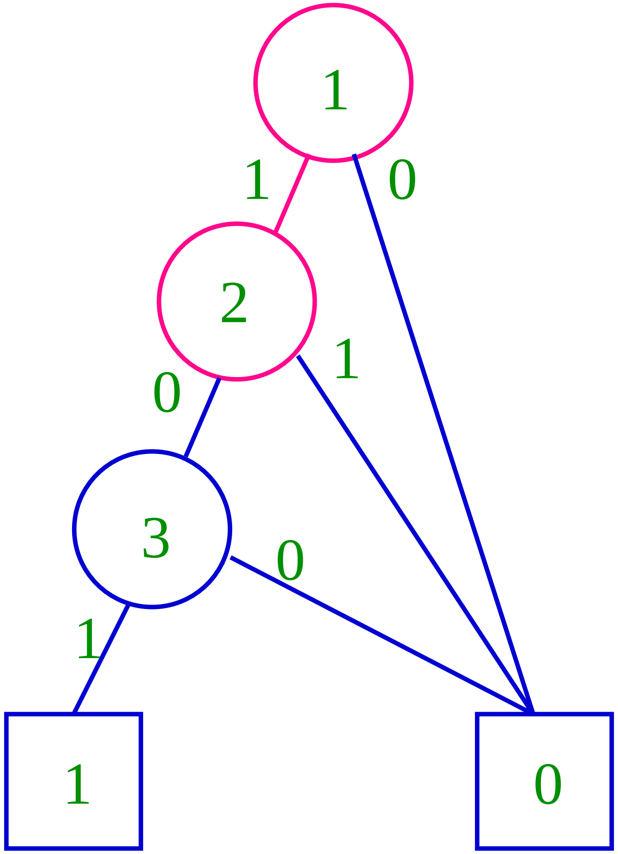
$x_1 \cdot \overline{x_2} \cdot x_3$



$x_1 = 0$

# A Satisfy-one Example

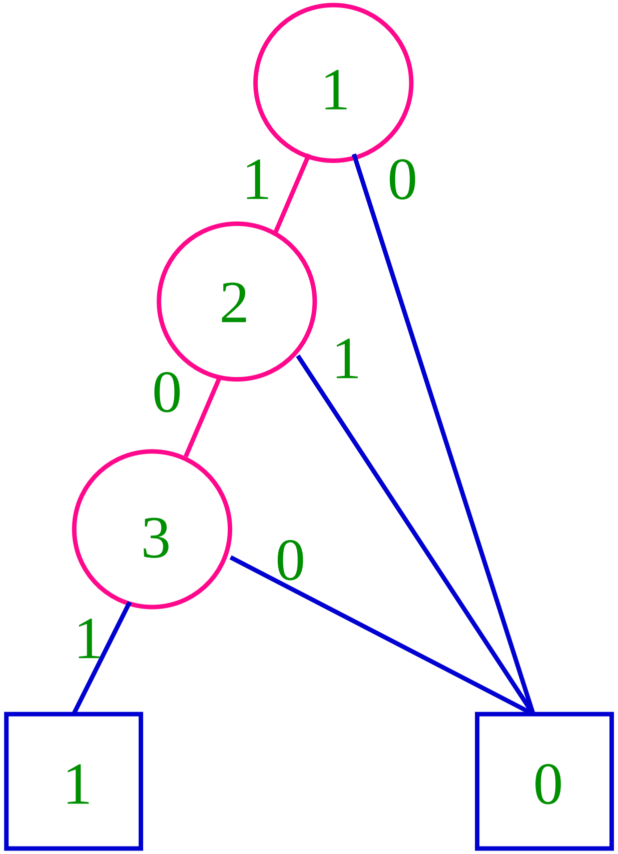
$x_1 \cdot \overline{x_2} \cdot x_3$



$x_1 = 1$

# A Satisfy-one Example

$x_1 \cdot \overline{x_2} \cdot x_3$

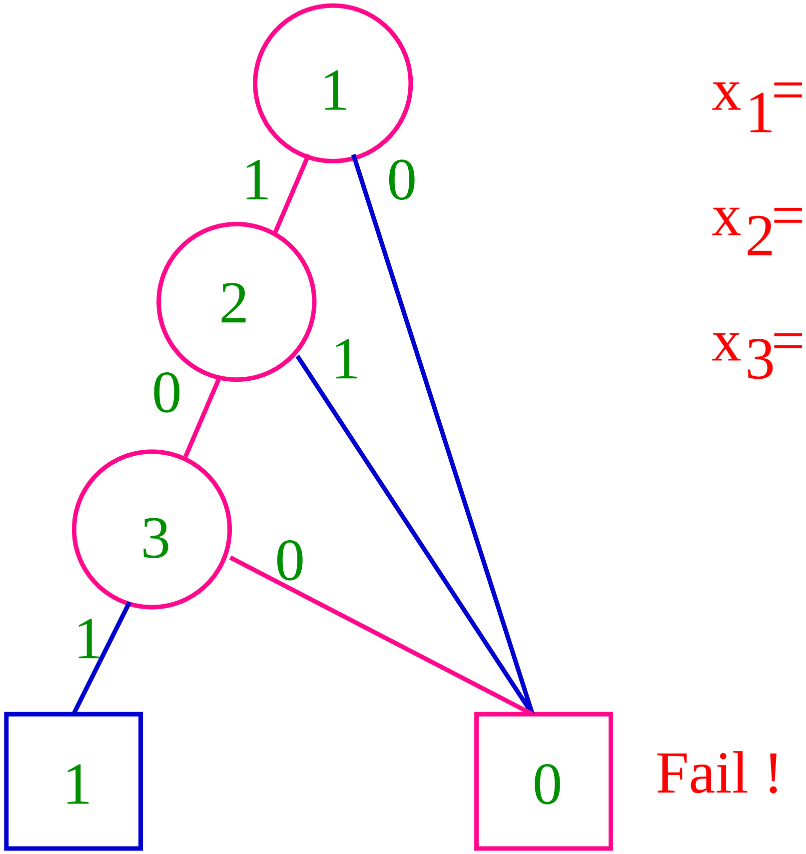


$x_1 = 1$

$x_2 = 0$

# A Satisfy-one Example

$x_1 \cdot \overline{x_2} \cdot x_3$



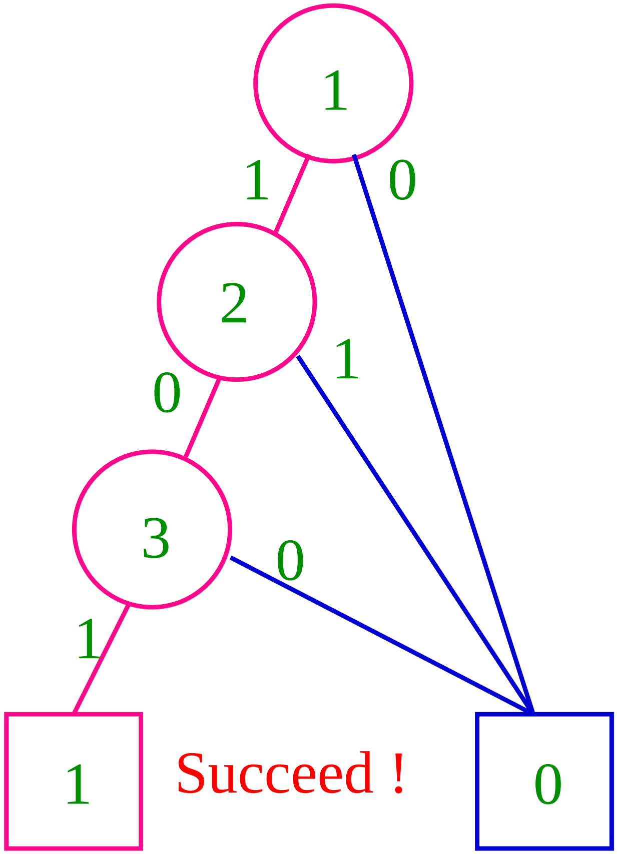
$x_1 = 1$

$x_2 = 0$

$x_3 = 0$

# A Satisfy-one Example

$x_1 \cdot \overline{x_2} \cdot x_3$



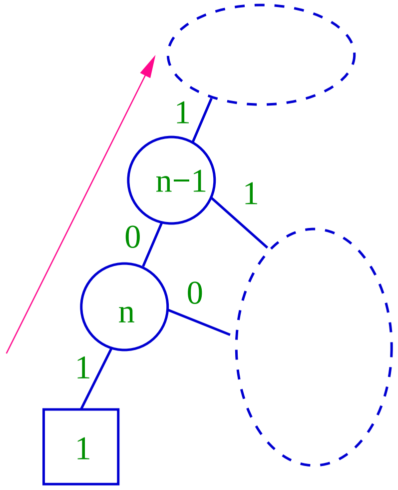
$x_1 = 1$

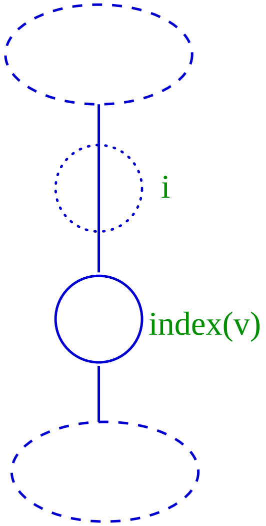
$x_2 = 0$

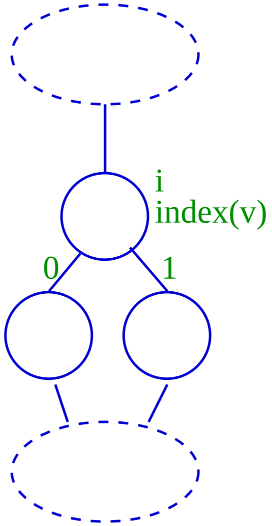
$x_3 = 1$

# Satisfy-all

```
procedure Satisfy-all(i: integer; v: vertex; x: array[1..n] of integer):
begin
    if value(v) = 0 then return;
    if i = n + 1 and value(v) = 1
    then begin
            Print element x[1],. . .,x[n];
            return;
    end;
    if index(v) > i
    then begin
            x[i] := 0; Satisfy-all(i + 1, v, x);
            x[i] := 1; Satisfy-all(i + 1, v, x);
    end
    else begin
            x[i] := 0; Satisfy-all(i + 1, low(v), x);
            x[i] := 1; Satisfy-all(i + 1, high(v), x);
    end
end;
```







# Satisfy-count

- The procedure *Satisfy-count* computes a value $\alpha_v$ to each vertex $v$ in the graph according to the following recursive formula:
  - If $v$ is a terminal vertex: $\alpha_v = value(v)$.
  - If $v$ is a nonterminal vertex:

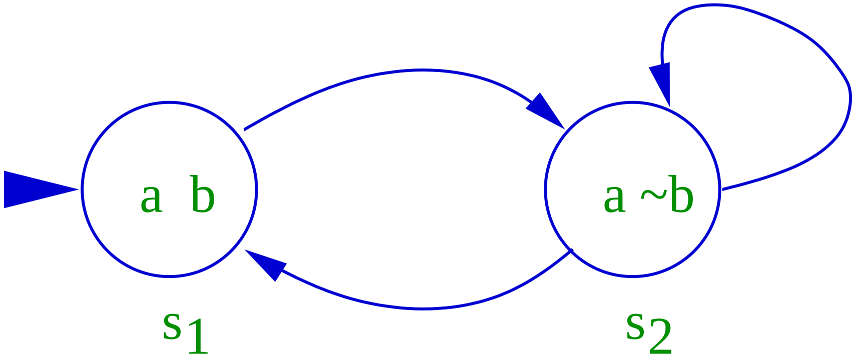
$$\alpha_v = \alpha_{low(v)} \cdot 2^{index(low(v)) - index(v)} + \alpha_{high(v)} \cdot 2^{index(high(v)) - index(v)}$$

- Once we have computed these values for a graph with root $v$, we compute the size of the satisfying set as

$$|S_f| = \alpha_v \cdot 2^{index(v) - 1}$$

# Kripke Structures

- Given a set of atomic propositions $AP$, a Kripke structure $M$ is a four tuple $(S, S_0, R, L)$:
  - $S$ is a finite set of states.
  - $S_0 \subseteq S$ is the set of initial states.
  - $R \subseteq S \times S$ is a transition relation that must be total.
  - $L : S \to 2^{AP}$ is a function that labels each state with the set of atomic propositions true in that state.



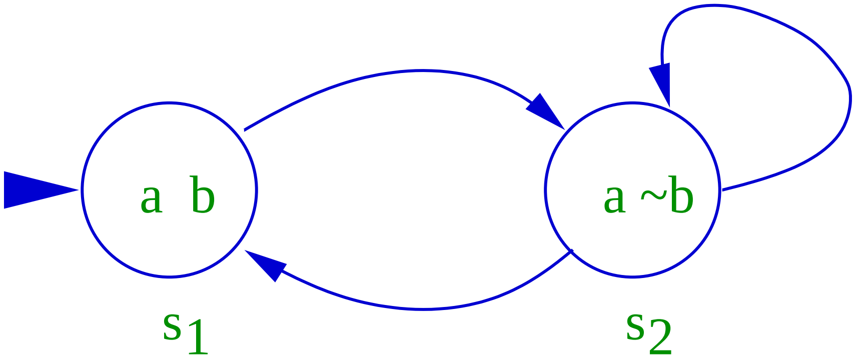
# First Order Representations

- The initial states can be represented by the formula:

$$(a \wedge b)$$

- The transitions can be represented by the formula:

$$\begin{aligned}&(a \wedge b \wedge a' \wedge \neg b') && \vee\\ &(a \wedge \neg b \wedge a' \wedge \neg b') && \vee\\ &(a \wedge \neg b \wedge a' \wedge b')\end{aligned}$$



# OBDD Representations

- Use $x_1, x_2, x_3, x_4$ to represent $a, b, a', b'$ respectively.
- The characteristic function of initial states:

$$(a \wedge b)$$

becomes

$$(x_1 \cdot x_2)$$

# OBDD Representations (cont.)

- The characteristic function of transitions:

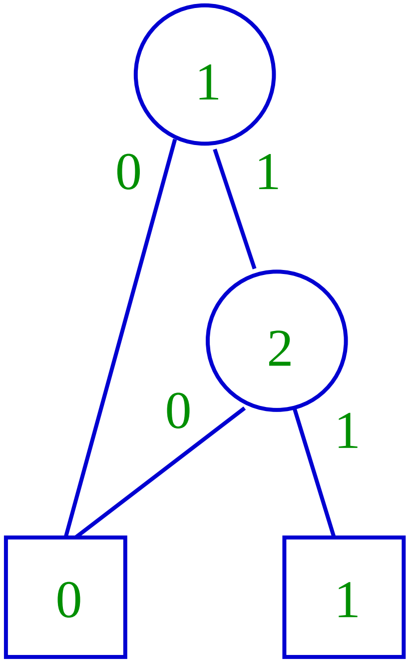
$$\begin{array}{ll} (a \wedge b \wedge a' \wedge \neg b') & \vee \\ (a \wedge \neg b \wedge a' \wedge \neg b') & \vee \\ (a \wedge \neg b \wedge a' \wedge b') & \end{array}$$

  becomes

$$\begin{array}{ll} (x_1 \cdot x_2 \cdot x_3 \cdot \bar{x}_4) & + \\ (x_1 \cdot \bar{x}_2 \cdot x_3 \cdot \bar{x}_4) & + \\ (x_1 \cdot \bar{x}_2 \cdot x_3 \cdot x_4) & \end{array}$$

# OBDD Representations (cont.)

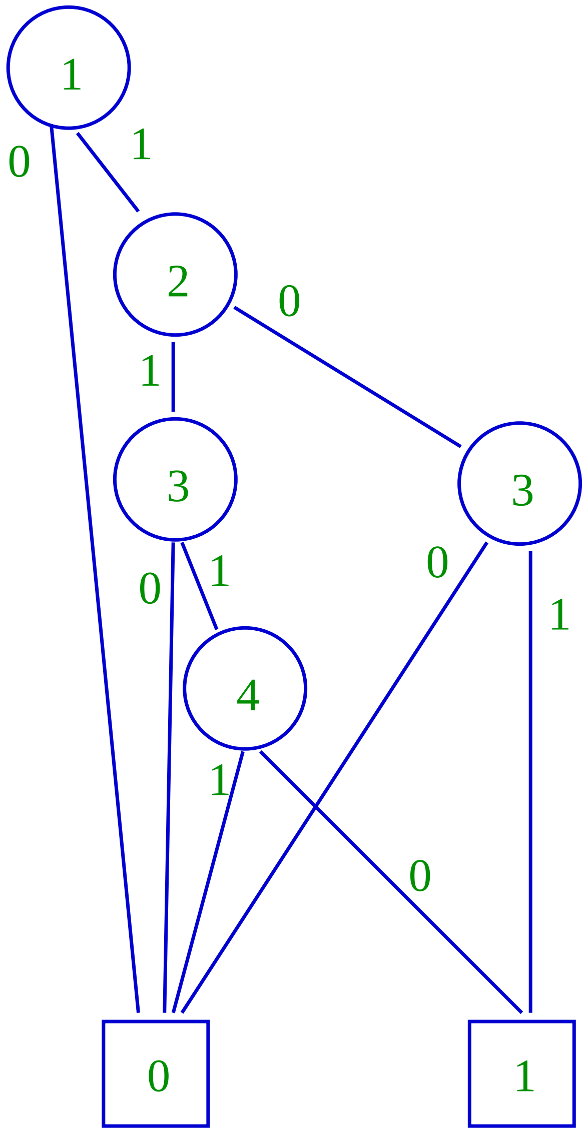
Initial states: $x_1 \cdot x_2$



# OBDD Representations (cont.)

Transitions:

$$(x_1 \cdot x_2 \cdot x_3 \cdot \bar{x}_4) \quad +$$
$$(x_1 \cdot \bar{x}_2 \cdot x_3 \cdot \bar{x}_4) \quad +$$
$$(x_1 \cdot \bar{x}_2 \cdot x_3 \cdot x_4)$$



# Summary

- OBDDs are representations of Boolean functions with
  - canonical forms, and
  - reasonable size.
- Transition systems can be encoded in Boolean functions and thus representable in OBDDs.
- Symbolic model checking becomes possible with OBDDs.

# Constant Functions

- Lemma 3: The reduced function graph $G$ denoting the constant function $0/1$ must consist of a single terminal vertex with value 0/1.

# Constant Functions (cont.)

- Let $G$ be a reduced graph denoting the constant function $\mathbf{0}$.
- $G$ cannot contain terminal vertices having value $1$.
- Suppose $G$ contains at least one nonterminal vertices.
    - There must be a nonterminal vertex $v$ where both $low(v)$ and $high(v)$ are terminal vertices. Thus we have $value(low(v)) = value(high(v))$.
    - Either (1) $low(v)$ and $high(v)$ are distinct, in which case $sub(G_f, low(v)) \sim sub(G_f, high(v))$ or (2) they are identical, in which case $low(v) = high(v)$.
    - In either case, $G_f$ would not be a reduced function graph.
- So, $G$ consists of a single terminal vertex with value $0$.

# Recall: Canonical Form

- Theorem: For any Boolean function $f$, there is a unique (up to isomorphism) reduced function graph denoting $f$ and any other function graph denoting $f$ contains more vertices.

# Proof of Canonical Form

- The proof proceeds by induction on the size of $I_f$
- Case 1: $|I_f| = 0$
  - The proof comes directly from Lemma 3.

# Proof of Canonical Form (cont.)

- Suppose that the theorem holds for any function $g$ having $|I_g| < k$.
- Consider an arbitrary function $f$ such that $|I_f| = k$, where $k > 0$.
- Let $i$ be the minimum value in $I_f$,.
- Define $f_0$ and $f_1$ as $f|_{x_i=0}$ and $f|_{x_i=1}$ respectively.
- $|I_{f_0}| < k$ and $|I_{f_1}| < k$ and hence $f_0$ and $f_1$ are represented by unique reduced function graphs $G_{f_0}$ and $G_{f_1}$ respectively.

# Proof of Canonical Form (cont.)

- Let $G_f$ and $G'_f$ be reduced function graphs for $f$.
- Let $v \in V_{G_f}$ and $v' \in V_{G'_f}$ be nonterminal vertices such that $index(v) = index(v') = i$.
- $sub(G_f, v)$ and $sub(G'_f, v')$ both denote $f$.
- $sub(G_f, low(v))$ and $sub(G'_f, low(v'))$ both denote $f_0$ and hence $sub(G_f, low(v)) \sim_{\sigma_0} sub(G'_f, low(v'))$ for some mapping $\sigma_0$.
- Similarly, $sub(G_f, high(v))$ and $sub(G'_f, high(v'))$ both denote $f_1$ and hence $sub(G_f, high(v)) \sim_{\sigma_1} sub(G'_f, high(v'))$ for some mapping $\sigma_1$.

# Proof of Canonical Form (cont.)

- We define a mapping $\sigma$ as

$$\sigma(u) = \begin{array}{ll} v', & u = v, \\ \sigma_0(u), & u \in V_{sub(G_f, low(v))} \\ \sigma_1(u), & u \in V_{sub(G_f, high(v))} \end{array}$$

- Claim 1: $\sigma$ is well-defined.
  - This comes from Claim 2 and Claim 3.

# Proof of Canonical Form (cont.)

- Claim 2: There is no conflict in $\sigma$.
  - If $u \in V_{sub(G_f, low(v))}$ and $u \in V_{sub(G_f, high(v))}$, then $sub(G'_f, \sigma_0(u)) \sim sub(G'_f, \sigma_1(u))$.
  - Since $G'_f$ contains no isomorphic subgraphs, this can only hold if $\sigma_0(u) = \sigma_1(u)$, and hence there is no conflict in the definition of $\sigma$.
- Claim 3: $\sigma$ must be one-to-one.
  - If there are distinct vertices $u_1$ and $u_2$ in $G_f$ having $\sigma(u_1) = \sigma(u_2)$, then $sub(G_f, u_1) \sim sub(G_f, u_2)$ and hence $G$ is not reduced.

# Proof of Canonical Form (cont.)

- Claim 4: $sub(G_f, v) \sim_\sigma sub(G'_f, v')$, $r(G_f) = v$, and $r(G'_f) = v'$.
  - We have shown $\sigma$ is a well-defined mapping.
  - Suppose there is some vertex $u$ with $index(u) = j < i$ such that there is no other vertex $w$ having $j < index(w) < i$.
  - $f$ does not depend on $x_j$ and hence $sub(G, low(u))$ and $sub(G, high(u))$ both define $f$.
  - The above implies $low(u) = high(u) = v$, i.e., $G$ is not reduced.
  - Hence $r(G) = v$.

# Proof of Canonical Form (cont.)

- Claim 5: Of all the graphs denoting a particular function, only the reduced graph has a minimum number of vertices.