Temporal Logic Model Checking

(Based on [Clarke et al. 1999])

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About Temporal Logic

- Temporal logic is a formalism for describing temporal ordering (or dependency) between occurrences of "events" (represented by propositions).
- It provides such expressive features by introducing temporal/modal operators into classic logic.
- These temporal operators usually do not explicitly mention time points.
- There are two principal views of the structure of time:
  - linear-time: occurrences of events form a sequence
  - branching-time: occurrences of events form a tree

Outline

- Temporal Logics
  - CTL* (generalized Computation Tree Logic)
  - CTL (Computation Tree Logic; subset of CTL*)
  - LTL (Linear Temporal Logic; subset of CTL*)
- Fairness
- Algorithmic Temporal Logic Verification
  - CTL Model Checking
  - LTL Model Checking
  - CTL* Model Checking

CTL*

- CTL* formulae describe properties of a computation tree (generated from a Kripke structure).
- They are composed of *path quantifiers* and *temporal operators*.
- Path quantifiers:
  - **E** (for some path)
  - **A** (for all paths)
- Temporal operators:
  - **X** (next)
  - **F** (eventually or sometime in the future)
  - **G** (always or globally)
  - **U** (until)
  - **R** (release)

Syntax of CTL*

- Let AP be a set of atomic propositions.
- The syntax of state formulae:
  - If $p \in AP$, then p is a state formula.
  - If f_1 and f_2 are state formulae, then so are $\neg f_1$, $f_1 \vee f_2$ and $f_1 \wedge f_2$.
  - If g is a path formula, then $\mathbf{E}g$ and $\mathbf{A}g$ are state formulae.
- The syntax of path formulae:
  - If f is a state formula, then f is also a path formula.
  - If g_1 and g_2 are path formulae, then so are $\neg g_1$, $g_1 \vee g_2$, $g_1 \wedge g_2$, $\mathbf{X}g_1$, $\mathbf{F}g_1$, $\mathbf{G}g_1$, $g_1 \ \mathbf{U} \ g_2$, and $g_1 \ \mathbf{R} \ g_2$.
- CTL* is the set of state formulae generated by the above rules.

Example CTL* Formulae

- Formula: $\mathbf{AG}(\mathit{Req} \rightarrow \mathbf{AF}\mathit{Ack})$.
  Intended meaning: every request will eventually be granted.
- Formula: $\mathbf{AG}(\mathbf{EF}\mathit{Restart})$.
  Intended meaning: it is always possible at any time to get to the *Restart* state.

Kripke Structures

- Let AP be a set of atomic propositions.
- A *Kripke structure* M over AP is a tuple $\langle S, S_0, R, L \rangle$:
  - S is a finite set of states,
  - $S_0 \subseteq S$ is the set of initial states,
  - $R \subseteq S \times S$ is a total transition relation, and
  - $L : S \rightarrow 2^{AP}$ is a function labeling each state with a subset of propositions (which are true in that state).
- A *computation* or *path* π of M from a state s is an infinite sequence $s_0, s_1, s_2, \cdots$ of states such that $s_0 = s$ and $(s_i, s_{i+1}) \in R$, for all $i \geq 0$.
- In the sequel, π^i denotes the *suffix* of π starting at s_i.

Semantics of CTL*

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- When f is a state formula, $M, s \models f$ means that f holds at state s in the Kripke structure M.
- When f is a path formula, $M, \pi \models f$ means that f holds along the path π in the Kripke structure M.
- Assuming that f_1 and f_2 are state formulae and g_1 and g_2 are path formulae, the semantics of CTL* is as follows:
  - $M, s \models p \iff p \in L(s)$
  - $M, s \models \neg f_1 \iff M, s \not\models f_1$
  - $M, s \models f_1 \vee f_2 \iff M, s \models f_1$ or $M, s \models f_2$
  - $M, s \models f_1 \wedge f_2 \iff M, s \models f_1$ and $M, s \models f_2$
  - $M, s \models \mathbf{E}g_1 \iff$ for some path π from s, $M, \pi \models g_1$
  - $M, s \models \mathbf{A}g_1 \iff$ for every path π from s, $M, \pi \models g_1$

Semantics of CTL* (cont.)

- The semantics of CTL* (cont.):
  - $M, \pi \models f_1 \iff s$ is the first state of π and $M, s \models f_1$
  - $M, \pi \models \neg g_1 \iff M, \pi \not\models g_1$
  - $M, \pi \models g_1 \vee g_2 \iff M, \pi \models g_1$ or $M, \pi \models g_2$
  - $M, \pi \models g_1 \wedge g_2 \iff M, \pi \models g_1$ and $M, \pi \models g_2$
  - $M, \pi \models \mathbf{X} g_1 \iff M, \pi^1 \models g_1$
  - $M, \pi \models \mathbf{F} g_1 \iff$ for some $k \geq 0$, $M, \pi^k \models g_1$
  - $M, \pi \models \mathbf{G} g_1 \iff$ for all $i \geq 0$, $M, \pi^i \models g_1$
  - $M, \pi \models g_1 \mathbf{U} g_2 \iff$ for some $k \geq 0$, $M, \pi^k \models g_2$ and, for all $0 \leq j < k, M, \pi^j \models g_1$
    (g_1 remains *true* until g_2 becomes *true*, which eventually happens.)
  - $M, \pi \models g_1 \mathbf{R} g_2 \iff$ for all $j \geq 0$, if for every $i < j$, $M, \pi^i \not\models g_1$, then $M, \pi^j \models g_2$
    (Only after g_1 becomes *true*, g_2 may become *false*.)

Minimalistic CTL*

- The operators $\vee$, $\neg$, **X**, **U**, and **E** are sufficient to express any other CTL* formula (in an equivalent way).
- In particular,
  - $\mathbf{F}f = \mathit{true}\ \mathbf{U}\ f$
  - $\mathbf{G}f = \neg\mathbf{F}\neg f$
  - $f\ \mathbf{R}\ g = \neg(\neg f\ \mathbf{U}\ \neg g)$
  - $\mathbf{A}f = \neg\mathbf{E}\neg f$
- $\neg(\neg f\ \mathbf{U}\ \neg g)$ says that it is not the case that in some state g becomes *false* and until then f has never been *true*.
- This is the same as saying that only after f becomes *true*, g may become *false* (or f "releases" g), namely $f\ \mathbf{R}\ g$.

CTL and LTL

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- CTL and LTL are restricted subsets of CTL*.
- CTL is a branching-time logic, while LTL is linear-time.
- In CTL, each temporal operator **X**, **F**, **G**, **U**, or **R** must be immediately preceded by a path quantifier **E** or **A**.
- The syntax of path formulae in CTL is more restricted:
  - If f_1 and f_2 are state formulae, then $\mathbf{X}f_1$, $\mathbf{F}f_1$, $\mathbf{G}f_1$, $f_1 \mathbf{~U~} f_2$, and $f_1 \mathbf{~R~} f_2$ are path formulae.
- The syntax of state formulae remains the same:
  - If $p \in AP$, then p is a state formula.
  - If f_1 and f_2 are state formulae, then so are $\neg f_1$, $f_1 \vee f_2$ and $f_1 \wedge f_2$.
  - If g is a path formula, then $\mathbf{E}g$ and $\mathbf{A}g$ are state formulae.

CTL and LTL (cont.)

- LTL consists of formulae that have the form $\mathbf{A}f$, where f is a path formula in which atomic propositions are the only permitted state formulae.
- The syntax of path formulae in LTL is as follows:
  - If $p \in AP$, then p is a path formula.
  - If g_1 and g_2 are path formulae, then so are $\neg g_1$, $g_1 \vee g_2$, $g_1 \wedge g_2$, $\mathbf{X} g_1$, $\mathbf{F} g_1$, $\mathbf{G} g_1$, $g_1 \mathbf{U} g_2$, and $g_1 \mathbf{R} g_2$.

Expressive Powers

- CTL, LTL, and CTL* have distinct expressive powers.
- Some discriminating examples:
  - $\mathbf{A}(\mathbf{FG}p)$ in LTL, not expressible in CTL.
  - $\mathbf{AG}(\mathbf{EF}p)$ in CTL, not expressible in LTL.
  - Both $\mathbf{A}(\mathbf{FG}p)$ and $\mathbf{AG}(\mathbf{EF}p)$ are expressible in CTL*.
- So, CTL* is strictly more expressive than CTL and LTL, the two of which are incomparable.

Fair Kripke Structures

- A fair Kripke structure is a 4-tuple $M = (S, R, L, F)$, where S, L, and R are defined as before and $F \subseteq 2^S$ is a set of fairness constraints. (Generalized Büchi acceptance conditions)
- Let $\pi = s_0, s_1, \ldots$ be a path in M.
- Define $\inf(\pi) = \{s \mid s = s_i \text{ for infinitely many } i\}$.
- We say that π is *fair* iff, for every $P \in F$, $\inf(\pi) \cap P \neq \emptyset$.

Fair Semantics

- We write $M, s \models_F f$ to indicate that the state formula f is true in state s of the fair Kripke structure M.
- $M, \pi \models_F g$ indicates that the path formula g is true along the path π in M.
- Only the following semantic rules are different from the original ones:
  - $M, s \models_F p \iff$ there exists a fair path starting from s and $p \in L(s)$.
  - $M, s \models_F \mathbf{E}g_1 \iff$ there exists a fair path π starting from s s.t. $M, \pi \models_F g_1$.
  - $M, s \models_F \mathbf{A}g_1 \iff$ for every fair path π starting from s, $M, \pi \models_F g_1$.

CTL Model Checking

- Let $M = (S, R, L)$ be a Kripke structure.
- We want to determine which states in S satisfy the CTL formula f.
- The algorithm will operate by labelling each state s with the set $label(s)$ of sub-formulae of f which are true in s.
  - Initially, $label(s)$ is just $L(s)$.
  - During the i-th stage, sub-formulae with $i - 1$ nested CTL operators are processed.
  - When a sub-formula is processed, it is added to the labelling of each state in which it is true.
  - Once the algorithm terminates, we will have that $M, s \models f$ iff $f \in \mathrm{label}(s)$.

Handling CTL Operators

- There are ten basic CTL temporal operators: **AX** and **EX**, **AF** and **EF**, **AG** and **EG**, **AU** and **EU**, and **AR** and **ER**.
- All these operators can be expressed in terms of **EX**, **EU**, and **EG**:
  - $\mathbf{AX}f = \neg\mathbf{EX}\neg f$
  - $\mathbf{EF}f = \mathbf{E}[\mathit{true}\ \mathbf{U}\ f]$
  - $\mathbf{AF}f = \neg\mathbf{EG}\neg f$
  - $\mathbf{AG}f = \neg\mathbf{EF}\neg f$
  - $\mathbf{A}[f\ \mathbf{U}\ g] = \neg\mathbf{E}[\neg g\ \mathbf{U}\ (\neg f \wedge \neg g)] \wedge \neg\mathbf{EG}\neg g$
    (This case is less obvious and will be proven next.)
  - $\mathbf{A}[f\ \mathbf{R}\ g] = \neg\mathbf{E}[\neg f\ \mathbf{U}\ \neg g]$ (from $f\ \mathbf{R}\ g = \neg(\neg f\ \mathbf{U}\ \neg g)$)
  - $\mathbf{E}[f\ \mathbf{R}\ g] = \neg\mathbf{A}[\neg f\ \mathbf{U}\ \neg g]$

The Case of AU

- To see why $\mathbf{A}[f\ \mathbf{U}\ g] = \neg\mathbf{E}[\neg g\ \mathbf{U}\ (\neg f \wedge \neg g)] \wedge \neg\mathbf{EG}\neg g$, let us introduce yet another temporal operator $\mathbf{W}$ (wait-for).
- Let $f\ \mathbf{W}\ g = f\ \mathbf{U}\ g \vee \mathbf{G}f$.
- It can be shown that
  - $f\ \mathbf{U}\ g = (f\ \mathbf{W}\ g) \wedge \mathbf{F}g$
  - $\neg(f\ \mathbf{W}\ g) = \neg g\ \mathbf{U}\ (\neg f \wedge \neg g)$.
- Proof of the rewriting for $\mathbf{A}[f\ \mathbf{U}\ g]$:

$$
\begin{array}{rl}
 & \mathbf{A}[f\ \mathbf{U}\ g] \\
= & \neg\mathbf{E}\neg(f\ \mathbf{U}\ g) \\
= & \neg\mathbf{E}\neg((f\ \mathbf{W}\ g) \wedge \mathbf{F}g) \\
= & \neg\mathbf{E}(\neg(f\ \mathbf{W}\ g) \vee \neg\mathbf{F}g) \\
= & \neg(\mathbf{E}\neg(f\ \mathbf{W}\ g) \vee \mathbf{E}\neg\mathbf{F}g) \\
= & \neg(\mathbf{E}[\neg g\ \mathbf{U}\ (\neg f \wedge \neg g)] \vee \mathbf{EG}\neg g) \\
= & \neg\mathbf{E}[\neg g\ \mathbf{U}\ (\neg f \wedge \neg g)] \wedge \neg\mathbf{EG}\neg g
\end{array}
$$

CTL Model Checking: AP, $\neg$, $\vee$, EX

So, for CTL model checking, it suffices to handle the following six cases: *atomic proposition*, $\neg$, $\vee$, **EX**, **EU**, and **EG**.

- Atomic propositions are handled at the beginning of the algorithm (by the initial setting $label(s) = L(s)$).
- For $\neg f$, we label those states that are not labelled by f.
- For $f_1 \vee f_2$, we label any state that is labelled either by f_1 or by f_2.
- For $\mathbf{EX} f$, we label every state that has some successor labelled by f.

CTL Model Checking: EU

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- To handle formulae of the form $\mathbf{E}[f_1 \ \mathbf{U} \ f_2]$, we follow these steps:
  - Find all states that are labelled with f_2.
  - Work backward using the converse of the transition relation R and find all states that can be reached by a path in which *each state* is labelled with f_1.
  - Label all such states by $\mathbf{E}[f_1 \ \mathbf{U} \ f_2]$.
- This requires time $O(|S| + |R|)$.

CTL Model Checking: EU (cont.)

IM NTU

procedure $CheckEU(f_1, f_2)$
  $T := \{s \mid f_2 \in label(s)\};$
  for all $s \in T$ **do** $label(s) := label(s) \cup \{\mathbf{E}[f_1 \ \mathbf{U}\ f_2]\};$
  while $T \neq \emptyset$ **do**
    choose $s \in T;$
    $T := T \setminus \{s\};$
    for all t **s.t.** $R(t, s)$ **do**
      if $\mathbf{E}[f_1 \ \mathbf{U}\ f_2] \notin label(t)$ **and** $f_1 \in label(t)$ **then**
        $label(t) := label(t) \cup \{\mathbf{E}[f_1 \ \mathbf{U}\ f_2]\};$
        $T := T \cup \{t\};$
      end if;
    end for all;
  end while;
end procedure;

CTL Model Checking: EG

- To handle formulae of the form $\mathbf{EG}f$, we need the following lemma:

  Let $M' = (S', R', L')$, where $S' = \{s \in S \mid M, s \models f\}$.
  $M, s \models \mathbf{EG}f$ iff the following two conditions hold:

  1. $s \in S'$.
  2. There exists a path in M' that leads from s to some node t in a *nontrivial* strongly connected component (SCC) C of the graph (S', R').

- Note: an SCC is nontrivial if either it contains at least two nodes or it contains only one node with a self loop.

CTL Model Checking: EG(cont.)

IM NTU

- With the lemma, we can handle $\mathbf{EG}f$ by the following steps:
  1. Construct the restricted Kripke structure M'.
  2. Partition the (S', R') into SCCs. (Complexity: $O(|S'| + |R'|)$).
  3. Find those states that belong to nontrivial components.
  4. Work backward using the converse of R' and find all of those states that can be reached by a path in which each state is labelled with f. (Complexity: $O(|S| + |R|)$)

CTL Model Checking: EG (cont.)

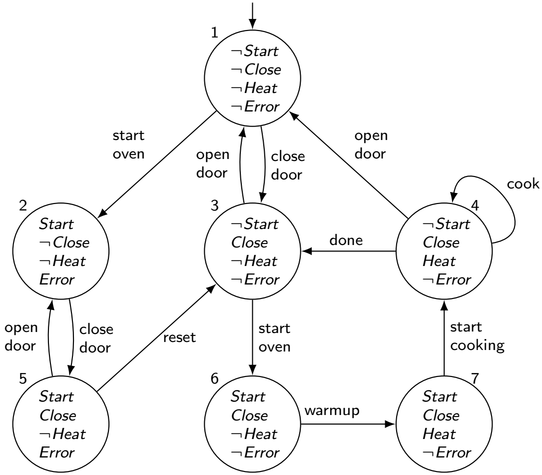
procedure $CheckEG(f)$
  $S' := \{s \mid f \in label(s)\}$;
  $SCC := \{C \mid C \text{ is a nontrivial SCC of S}'\}$;
  $T := \bigcup_{C \in SCC} \{s \mid s \in C\}$;
  for all $s \in T$ **do** $label(s) := label(s) \cup \{\mathbf{EG}f\}$;
  while $T \neq \emptyset$ **do**
    choose $s \in T$;
    $T := T \setminus \{s\}$;
    for all t **s.t.** $t \in S'$ **and** $R(t, s)$ **do**
      if $\mathbf{EG}f \notin label(t)$ **and** $f \in label(t)$ **then**
        $label(t) := label(t) \cup \{\mathbf{EG}f\}$;
        $T := T \cup \{t\}$;
      end if;
    end for all;
  end while; **end procedure**;

CTL Model Checking (cont.)

- We successively apply the state-labelling algorithm to the sub-formulae of f, starting with the shortest, most deeply nested, and work outward to include the whole formula.
- By proceeding in this manner, we guarantee that whenever we process a sub-formula of f all its sub-formulae have already been processed.
- There are at most $|f|$ sub-formulae, and each formula takes at most $O(|S| + |R|)$ time.
- The complexity of this algorithm is $O(|f| \cdot (|S| + |R|))$.

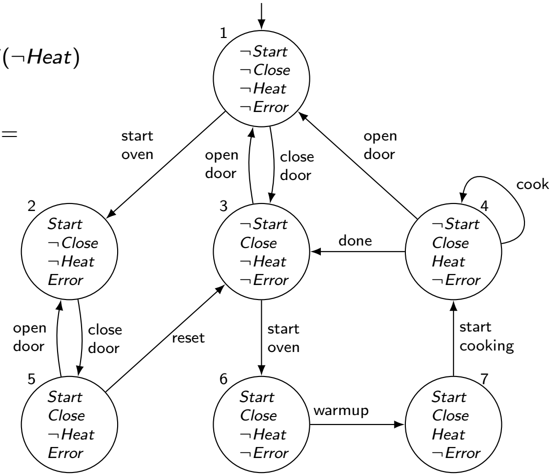
An Example

We want to check CTL formula $\mathbf{AG}(Start \rightarrow \mathbf{AF}Heat)$, or $\neg\mathbf{E}[true\ \mathbf{U}\ (Start\ \wedge\ \mathbf{EG}\neg Heat)]$.



An Example (cont.)

Spec.: $\neg \mathbf{E}[\mathit{true}\ \mathbf{U}\ (\mathit{Start}\ \wedge \mathbf{EG}\neg \mathit{Heat})]$
$S(\mathit{Start}) = \{2, 5, 6, 7\}$
$S(\neg \mathit{Heat}) = \{1, 2, 3, 5, 6\}$
Find SCC $\{\{1, 2, 3, 5\}\}$ in $S' = S(\neg \mathit{Heat})$
$S(\mathbf{EG}\neg \mathit{Heat}) = \{1, 2, 3, 5\}$
$S(\mathit{Start} \wedge \mathbf{EG}\neg \mathit{Heat}) = \{2, 5\}$
$S(\mathbf{E}[\mathit{true}\ \mathbf{U}\ (\mathit{Start}\ \wedge \mathbf{EG}\neg \mathit{Heat}]) =$
$\{1, 2, 3, 4, 5, 6, 7\}$



Fairness Constraints

- Let $M = (S, R, L, F)$ be a fair Kripke structure.
- Let $F = \{P_1, \ldots, P_k\}$.
- We say that a SCC C is *fair* w.r.t F iff for each $P_i \in F$, there is a state $t_i \in (C \cap P_i)$.
- To handle formulae of the form $\mathbf{EG}f$ in a fair kripke structure, we need the following lemma:

  Let $M' = (S', R', L', F')$, where $S' = \{s \in S \mid M, s \models_F f\}$.
  $M, s \models_F EGf$ iff the following two conditions holds:

  1. $s \in S'$.
  2. There exists a path in S' that leads from s to some node t in a nontrivial fair strongly connected component of the graph (S', R').

Fairness Constraints

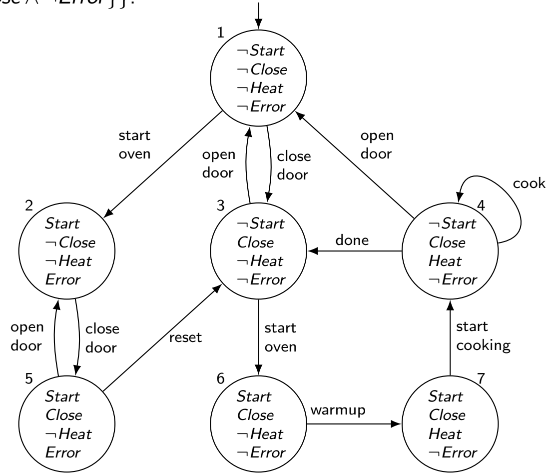
- We can create a *CheckFairEG* algorithm which is very similar to the *CheckEG* algorithm based on this lemma.
- The complexity of *CheckFairEG* is $O((|S| + |R|) \cdot |F|)$, since we have to check which SCC is fair.
- To check other CTL formulae, we introduce another proposition *fair* and stipulate that

$$M, s \models \mathit{fair} \text{ iff } M, s \models_F \mathbf{EG}\,\mathit{true}.$$

- $M, s \models_F p$, for some $p \in AP$, we check $M, s \models p \wedge \mathit{fair}$.
- $M, s \models_F \mathbf{EX} f$, we check $M, s \models \mathbf{EX}(f \wedge \mathit{fair})$.
- $M, s \models_F \mathbf{E}[f_1 \ \mathbf{U} \ f_2]$, we check $M, s \models \mathbf{E}[f_1 \ \mathbf{U} \ (f_2 \wedge \mathit{fair})]$.
- Overall complexity: $O(|f| \cdot (|S| + |R|) \cdot |F|)$.

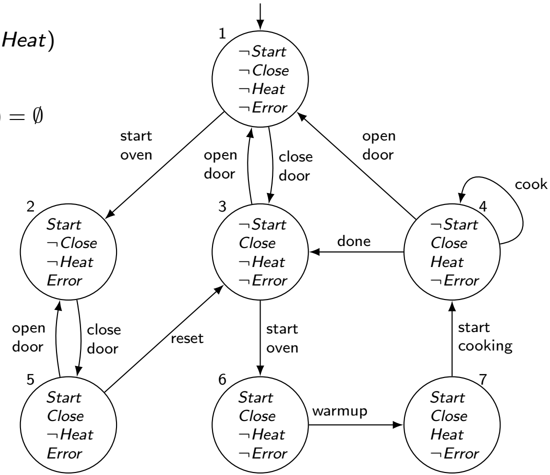
An Example

Assume $F = \{\{s \mid s \models Start \land Close \land \neg Error\}\}$.
We want to check CTL formula
$\mathbf{AG}(Start \rightarrow \mathbf{AF}\,Heat)$, or
$\neg\mathbf{E}[true\ \mathbf{U}\ (Start\ \land \mathbf{EG}\neg Heat)]$.



An Example (cont.)

Spec.: $\neg\mathbf{E}[\mathit{true}\ \mathbf{U}\ (\mathit{Start}\ \wedge \mathbf{EG}\neg\mathit{Heat})]$
$S(\mathit{Start}) = \{2, 5, 6, 7\}$
$S(\neg\mathit{Heat}) = \{1, 2, 3, 5, 6\}$
There is no fair SCC in $S' = S(\neg\mathit{Heat})$
$S(\mathbf{EG}\neg\mathit{Heat}) = \emptyset$
$S(\mathit{Start} \wedge \mathbf{EG}\neg\mathit{Heat}) = \emptyset$
$S(\mathbf{E}[\mathit{true}\ \mathbf{U}\ (\mathit{Start}\ \wedge \mathbf{EG}\neg\mathit{Heat})]) = \emptyset$



The LTL Model Checking Problem

- Let $M = (S, R, L)$ be a Kripke structure with $s \in S$.
- Let $\mathbf{A}g$ be an LTL formula (so, g is a restricted path formula).
- We want to check if $M, s \models \mathbf{A}g$.
- $M, s \models \mathbf{A}g$ iff $M, s \models \neg\mathbf{E}\neg g$.
- Therefore, it suffices to be able to check $M, s \models \mathbf{E}f$, where f is a restricted path formula.

Complexity of LTL Model Checking

- The problem is PSPACE-complete.
- We can more easily show this problem to be NP-hard by a reduction from the Hamiltonian path problem.
- Consider a directed graph $G = (V, A)$ where $V = \{v_1, v_2, \ldots, v_n\}$.
- Determining whether G has a directed Hamiltonian path is reducible to the problem of determining whether $M, s \models f$, where
  - M is a finite Kripke structure (constructed from G),
  - s is a state in M, and
  - f is the formula (using atomic propositions $p_1, \ldots, p_n$):

$$\mathbf{E}[\mathbf{F}p_1 \wedge \ldots \wedge \mathbf{F}p_n \wedge \mathbf{G}(p_1 \rightarrow \mathbf{XG}\neg p_1) \wedge \ldots \wedge \mathbf{G}(p_n \rightarrow \mathbf{XG}\neg p_n)].$$

Complexity of LTL Model Checking (cont.)

- The Kripke structure $M = (U, B, L)$ is obtained from $G = (V, A)$ as follows:
  - $U = V \cup \{u_1, u_2\}$ where $u_1, u_2 \notin V$.
  - $B = A \cup \{(u_1, v_i) \mid v_i \in V\} \cup \{(v_i, u_2) \mid v_i \in V\} \cup \{(u_2, u_2)\}$.
  - L is an assignment of propositions to states s.t.:
    - p_i is true in v_i for $1 \leq i \leq n$,
    - p_i is false in v_j for $1 \leq i, j \leq n$, $i \neq j$, and
    - p_i is false in u_1, u_2 for $1 \leq i \leq n$.
- Let s be u_1.
- $M, u_1 \models f$ iff there is a directed infinite path in M starting at u_1 that goes through every $v_i \in V$ exactly once and ends in the self loop at u_2.

LTL Model Checking

Here we introduce an algorithm by Lichtenstein and Pnueli.

- The algorithm is exponential in the length of the formula, but linear in the size of the state graph.
- It involves an implicit tableau construction.
- A tableau is a graph derived from the formula from which a model for the formula can be extracted iff the formula is satisfiable.
- To check whether M satisfies f, the algorithm composes the tableau and the Kripke structure and determines whether there exists a computation of the structure that is a path in the tableau.

Closure

- Like before, we need only deal with **X** and **U**.
- The closure $CL(f)$ of f contains formulae whose truth values can influence the truth value of f.
- It is the smallest set containing f and satisfying:
  - $\neg f_1 \in CL(f)$ iff $f_1 \in CL(f)$,
  - if $f_1 \wedge f_2 \in CL(f)$, then $f_1, f_2 \in CL(f)$,
  - if $\mathbf{X} f_1 \in CL(f)$, then $f_1 \in CL(f)$,
  - if $\neg \mathbf{X} f_1 \in CL(f)$, then $\mathbf{X} \neg f_1 \in CL(f)$,
  - if $f_1 \mathbf{U} f_2 \in CL(f)$, then $f_1, f_2, \mathbf{X}[f_1 \mathbf{U} f_2] \in CL(f)$.

Atom

- An *atom* is a pair $A = (s_A, K_A)$ with $s_A \in S$ and $K_A \subseteq CL(f) \cup AP$ s.t.:
  - for each proposition $p \in AP$, $p \in K_A$ iff $p \in L(s_A)$,
  - for every $f_1 \in CL(f)$, $f_1 \in K_A$ iff $\neg f_1 \notin K_A$,
  - for every $f_1 \vee f_2 \in CL(f)$, $f_1 \vee f_2 \in K_A$ iff $f_1 \in K_A$ or $f_2 \in K_A$,
  - for every $\neg \mathbf{X} f_1 \in CL(f)$, $\neg \mathbf{X} f_1 \in K_A$ iff $\mathbf{X} \neg f_1 \in K_A$,
  - for every $f_1 \ \mathbf{U} \ f_2 \in CL(f)$, $f_1 \ \mathbf{U} \ f_2 \in K_A$ iff $f_2 \in K_A$ or $f_1, \mathbf{X}[f_1 \ \mathbf{U} \ f_2] \in K_A$.
- Intuitively, an atom (s_A, K_A) is defined so that K_A is a maximal consistent set of formulae that are also consistent with the labelling of s_A.

Behavior Graph and Self-Fulfilling SCC

- A graph G is constructed with the set of atoms as the set of vertices.
- (A, B) is an edge of G iff
  - $(s_A, s_B) \in R$ and
  - for every formula $\mathbf{X} f_1 \in CL(f)$, $\mathbf{X} f_1 \in K_A$ iff $f_1 \in K_B$.
- A nontrivial SCC C of the graph G is said to be *self-fulfilling* iff for every atom A in C and for every $f_1 \ \mathbf{U} \ f_2 \in K_A$ there exists an atom B in C s.t. $f_2 \in K_B$.
- **Lemma**: $M, s \models \mathbf{E} f$ iff there exists an atom (s, K) in G s.t. $f \in K$ and there is a path in G from (s, K) to a self-fulfilling SCC.

Sketch of the Correctness Proof

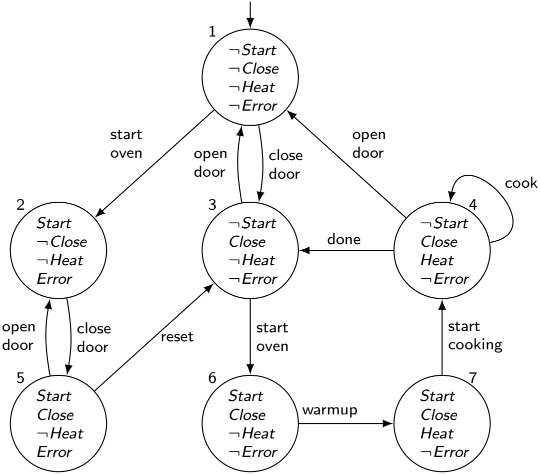
- A path ρ in G (generated from M and f) is an *eventuality sequence* if $f_1 \ \mathbf{U} \ f_2 \in K_A$ for some atom A on ρ, then there exists an atom B, reachable from A along π, such that $f_2 \in K_B$.
- Claim: $M, s \models \mathbf{E}f$ iff there exists an eventuality sequence starting from (s, K) such that $f \in K$.
  - $(\Leftarrow)$ If $\pi = s_0 (= s), s_1, s_2, \cdots$ corresponds to an eventuality sequence $(s, K) = (s_0, K_0), (s_1, K_1), \cdots$, then for every $g \in CL(f)$ and every $i \geq 0$, $\pi^i \models g$ iff $g \in K_i$.
  - $(\Rightarrow)$ For a path $\pi = s_0 (= s), s_1, s_2, \cdots$ such that $M, \pi \models f$, define $K_i = \{g \mid g \in CL(f) \text{ and } \pi^i \models g\}$, then $(s_0, K_0), (s_1, K_1), \cdots$ is an eventuality sequence.
- Claim: there exists an eventuality sequence starting from (s, K) iff there is a path in G from (s, K) to a self-fulfilling SCC.

The LTL Model Checking Algorithm

- Given a Kripke structure $M = (S, R, L)$, we want to check if $M, s \models \mathbf{E}f$, where f is a restricted path formula.
  - Construct the behavior graph $G = (V, E)$.
  - Find initial atom set $A = \{(s, K) \mid (s, K) \in V \wedge f \in K\}$.
  - Consider nontrivial self-fulfilling SCCs, traverse backward using the converse of E and mark all reachable states.
  - If any state in A is marked, $M, s \models \mathbf{E}f$ is true.
- Time complexity: $O((|S| + |R|) \cdot 2^{O(|f|)})$.
- For a fair Kripke structure $M' = (S', R', L', F')$, we should check if there exists any self-fulfilling and fair SCC.

An Example

We want to check LTL formula $\mathbf{A}[\neg \mathit{Heat}\ \mathbf{U}\ \mathit{Close}]$, or $\neg\mathbf{E}\neg[\neg \mathit{Heat}\ \mathbf{U}\ \mathit{Close}]$.



An Example (cont.)

- Let f denote $\neg Heat \ \mathbf{U}\ Close$.
- $CL(\neg f) = \{\neg f, f, \mathbf{X}f, \neg\mathbf{X}f, \mathbf{X}\neg f, Heat, \neg Heat, Close, \neg Close\}$.
- $\neg Close$ and $\neg Heat$ in states 1 and 2, so the possible "K" includes $\{\neg Close, \neg Heat, f, \mathbf{X}f\}, \{\neg Close, \neg Heat, \neg f, \mathbf{X}\neg f, \neg\mathbf{X}f\}$.
- $Close$ and $\neg Heat$ in states 3, 5 and 6, so the possible "K" includes $\{Close, \neg Heat, f, \mathbf{X}f\}, \{Close, \neg Heat, f, \mathbf{X}\neg f, \neg\mathbf{X}f\}$.
- $Close$ and $Heat$ in states 4 and 7, so the possible "K" includes $\{Close, Heat, f, \mathbf{X}f\}, \{Close, Heat, f, \mathbf{X}\neg f, \neg\mathbf{X}f\}$.

We can construct atoms using the states and the corresponding "K" and then build a graph based on those atoms.

Overview of CTL* Model Checking

- We will study an algorithm developed by Clarke, Emerson, and Sistla.
- The basic idea is to integrate the state labeling technique from CTL model checking into LTL model checking.
- The algorithm for LTL handles formula of the form $\mathbf{E}f$ where f is a restricted path formula.
- The algorithm can be extended to handle formulae in which f contains arbitrary state sub-formulae.

Handling CTL* Operators

Again, the operators $\neg, \vee, \mathbf{X}, \mathbf{U}$, and $\mathbf{E}$ are sufficient to express any other CTL* formula.

- $f \wedge g \equiv \neg(\neg f \vee \neg g)$
- $\mathbf{F} f \equiv \mathit{true}\ \mathbf{U}\ f$
- $\mathbf{G} f \equiv \neg \mathbf{F} \neg f$
- $f\ \mathbf{R}\ g \equiv \neg(\neg f\ \mathbf{U}\ \neg g)$
- $\mathbf{A} f \equiv \neg \mathbf{E} \neg f$

One Stage in CTL* Model Checking

- Let $\mathbf{E}f'$ be an "inner most" formula with $\mathbf{E}$.
- Assuming that the state sub-formulae of f' have already been processed and that state labels have been updated accordingly, proceed as follows:
  - If $\mathbf{E}f'$ is in CTL, then apply the CTL algorithm.
  - Otherwise, f' is a LTL path formula, then apply the LTL model checking algorithm.
  - In both cases, the formula is added to the labels of all states that satisfy it.
- If $\mathbf{E}f'$ is a sub-formula of a more complex CTL* formula, then the procedure is repeated with $\mathbf{E}f'$ replaced by a fresh AP.
- Note: each state sub-formula will be replaced by a fresh AP in both the labeling of the model and the formula.

Levels of State Sub-formulae

- The state sub-formulae of level i are defined inductively as follows:
  - Level 0 contains all atomic propositions.
  - Level $i + 1$ contains all state sub-formulae g s.t. all state sub-formulae of g are of level i or less and g is not contained in any lower level.
- Let g be a CTL* formula, then a sub-formula $\mathbf{E}h_1$ of g is *maximal* iff $\mathbf{E}h_1$ is not a strict sub-formula of any strict sub-formula $\mathbf{E}h$ of g.

State Sub-formulae (Examples)

- Consider the formula
  $\neg \mathbf{EF}(\neg \mathit{Close} \wedge \mathit{Start} \wedge \mathbf{E}(\mathbf{F}\mathit{Heat} \wedge \mathbf{G}\mathit{Error}))$.
- The levels of the state sub-formulae are:
  - Level 0: *Close*, *Start*, *Heat*, and *Error*
  - Level 1: $\mathbf{E}(\mathbf{F}\mathit{Heat} \wedge \mathbf{G}\mathit{Error})$ and $\neg \mathit{Close}$
  - Level 2: $\neg \mathit{Close} \wedge \mathit{Start}$
  - Level 3: $\neg \mathit{Close} \wedge \mathit{Start} \wedge \mathbf{E}(\mathbf{F}\mathit{Heat} \wedge \mathbf{G}\mathit{Error})$
  - Level 4: $\mathbf{EF}(\neg \mathit{Close} \wedge \mathit{Start} \wedge \mathbf{E}(\mathbf{F}\mathit{Heat} \wedge \mathbf{G}\mathit{Error}))$
  - Level 5: $\neg \mathbf{EF}(\neg \mathit{Close} \wedge \mathit{Start} \wedge \mathbf{E}(\mathbf{F}\mathit{Heat} \wedge \mathbf{G}\mathit{Error}))$
- Note: this is slightly different from [Clarke *et al.*].

CTL* Model Checking

- Let $M = (S, R, L)$ be a Kripke structure, f a CTL* formula, and g a state sub-formula of f of level i.
- The states of M have already been labelled correctly with all state sub-formulae of level smaller than i.
- In stage i, each such g is added to the labels of all states that make it true.
- For g a CTL* state formula, we proceed as follows:
  - If $g \in \text{AP}$, then g is in $label(s)$ iff it is in $L(s)$.
  - If $g = \neg g_1$, then g is in $label(s)$ iff g_1 is not in $label(s)$.
  - If $g = g_1 \vee g_2$, then g is added to $label(s)$ iff either g_1 or g_2 are in $label(s)$. (To reduce the number of levels, do analogously for $g_1 \wedge g_2$.)
  - If $g = \mathbf{E}g_1$ call the $CheckE(g)$ procedure.

$CheckE(g)$ Procedure

procedure $CheckE(g)$
  if g is a CTL formula **then**
    apply CTL model checking for g;
    return; // next formula or next stage
  end if;
  $g' := g[a_1/\mathbf{E}h_1, \ldots, a_k/\mathbf{E}h_k]$; // $\mathbf{E}h_i$'s are maximal sub-formulae
  for all $s \in S$
    for $i = 1, \ldots, k$ **do**
      if $\mathbf{E}h_i \in label(s)$ **then** $label(s) := label(s) \cup \{a_i\}$;
  end for all;
  apply LTL model checking for g';
  for all $s \in S$ **do**
    if $g' \in label(s)$ **then** $label(s) := label(s) \cup \{g\}$;
  end for all;
end procedure;

Complexity of the Algorithm for CTL*

- The complexity depends on the complexity of the algorithm for CTL and that for LTL.
- So, if the previous algorithms are used, the complexity is $O((|S| + |R|) \cdot 2^{O(|f|)})$.
- In real implementation, state sub-formulae need not be replaced by, but just need to be treated as, atomic propositions.