Suggested Solutions for Homework #1

(prepared by Jen-Feng Shih)

1. Determine, using *Truth Tables*, the validity of the following propositions: (30 points)

   (a) $(p \vee q \to r) \to (p \to r) \wedge (q \to r)$

   Solution. Let Φ_a denote $(p \vee q \to r) \to (p \to r) \wedge (q \to r)$.

p	q	r	$p \vee q$	$p \vee q \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	Φ_a
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

$\square$

   (b) $(p \to (q \to r)) \to (p \wedge q \to r)$

   Solution. Let Φ_b denote $(p \to (q \to r)) \to (p \wedge q \to r)$.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$p \wedge q \to r$	Φ_b
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

$\square$

   (c) $((p \to q) \to p) \to p$

   Solution. Let Φ_c denote $((p \to q) \to p) \to p$.

p	q	$p \to q$	$(p \to q) \to p$	Φ_c
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

$\square$

2. Prove, using *Natural Deduction*, the validity of the following sequents: (30 points)

   (a) $(p \to r) \wedge (q \to r) \vdash p \vee q \to r$

   Solution.

$$
\dfrac{\dfrac{}{(p \to r) \wedge (q \to r), p \vee q \vdash p \vee q}\,(Hyp) \qquad \alpha \qquad \beta}{\dfrac{(p \to r) \wedge (q \to r), p \vee q \vdash r}{(p \to r) \wedge (q \to r) \vdash p \vee q \to r}\,(\to I)}\,(\vee E)
$$

α:

$$
\dfrac{\dfrac{\dfrac{}{(p \to r) \wedge (q \to r), p \vee q, p \vdash (p \to r) \wedge (q \to r)}\,(Hyp)}{(p \to r) \wedge (q \to r), p \vee q, p \vdash p \to r}\,(\wedge E_1) \qquad \dfrac{}{(p \to r) \wedge (q \to r), p \vee q, p \vdash p}\,(Hyp)}{(p \to r) \wedge (q \to r), p \vee q, p \vdash r}\,(\to E)
$$

β:

$$
\dfrac{\dfrac{\dfrac{}{(p \to r) \wedge (q \to r), p \vee q, p \vdash (p \to r) \wedge (q \to r)}\,(Hyp)}{(p \to r) \wedge (q \to r), p \vee q, q \vdash q \to r}\,(\wedge E_2) \qquad \dfrac{}{(p \to r) \wedge (q \to r), p \vee q, q \vdash q}\,(Hyp)}{(p \to r) \wedge (q \to r), p \vee q, q \vdash r}\,(\to E)
$$

□

(b) $p \wedge q \to r \vdash p \to (q \to r)$

Solution.

$$
\dfrac{\dfrac{\dfrac{}{p \wedge q \to r, p, q \vdash p \wedge q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \wedge q \to r, p, q \vdash p}\,(Hyp) \qquad \dfrac{}{p \wedge q \to r, p, q \vdash q}\,(Hyp)}{p \wedge q \to r, p, q \vdash p \wedge q}\,(\wedge I)}{\dfrac{p \wedge q \to r, p, q \vdash r}{p \wedge q \to r, p \vdash q \to r}\,(\to I)}\,(\to E)}{p \wedge q \to r \vdash p \to (q \to r)}\,(\to I)
$$

□

(c) $\vdash ((p \to q) \to p) \to p$

Solution. (Jinn-Shu Chang)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to p, \neg p \vdash (p \to q) \to p}\,(Hyp) \qquad \alpha}{(p \to q) \to p, \neg p \vdash p}\,(\to E) \qquad \dfrac{}{(p \to q) \to p, \neg p \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p \vdash p \wedge \neg p}\,(\wedge I)}{(p \to q) \to p \vdash \neg\neg p}\,(\neg I)}{(p \to q) \to p \vdash p}\,(\neg\neg E)}{\vdash ((p \to q) \to p) \to p}\,(\to I)
$$

α:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \to q) \to q, \neg p, p, \neg q \vdash p}\,(Hyp) \qquad \dfrac{}{(p \to q) \to q, \neg p, p, \neg q \vdash \neg p}\,(Hyp)}{(p \to q) \to p, \neg p, p, \neg q \vdash p \wedge \neg p}\,(\wedge I)}{(p \to q) \to p, \neg p, p \vdash \neg\neg q}\,(\neg I)}{(p \to q) \to p, \neg p, p \vdash q}\,(\neg\neg E)}{(p \to q) \to p, \neg p \vdash p \to q}\,(\to I)
$$

□

3. Express each of the following requirements as a first-order formula. (40 points)

   (a) R is a functional binary relation. (Note: a binary relation R is *functional* if no element is related by R to more than one elements.)

   Solution.

$$\forall a, b, c(R(a, b) \land R(a, c) \to b = c)$$

□

   (b) In graph $G(V, E)$ a node, named u, has exactly two neighbors.

   Solution.

$$\exists a, b \in V(E(u, a) \land E(u, b) \land \neg(a = b) \land \forall c \in V(E(u, c) \to c = a \lor c = b))$$

□

   (c) a is the smallest prime number larger than b.

   Solution. Let div denote a binary predicate such that $div(x, y)$ is *true* if and only if x divides y.

$$\forall i(b < i < a \to \exists j(div(j, i) \land \neg(j = 1) \land \neg(j = i))) \land \forall k(div(k, a) \to k = 1 \lor k = a) \land (b < a)$$

□

   (d) Array A, with n elements, is cyclically sorted. (Note: an array is *cyclically sorted* if, by circular-shifting its elements, one can turn it into a sorted array.)

   Solution. We assume that the n elements are indexed from 0 through $n - 1$.

$$\begin{array}{lll} \exists k(& & (k = n - 1 \lor (0 \le k < n - 1 \land A[n-1] < A[0]) \\ & \land & \forall i(0 \le i < k \to A[i] \le A[i+1]) \\ & \land & \forall i(k \le i < n - 1 \to A[i] \le A[i+1]) \qquad) \end{array}$$

   Alternatively,

$$\exists k(\forall i(0 \le i < n \to A[(k + i) \bmod n] \le A[(k + i + 1) \bmod n]))$$

   where "$a \bmod n$" gives the remainder of a divided by n.

   Note: in pure logic terms, an array may be modeled as a function, so $A[i]$ would be written as $A(i)$. □