

Formal Logic: Propositional Logic

(Based on [Gallier 1986], [Goubault-Larrecq and Mackie 1997], and [Huth and Ryan 2004])

Yih-Kuen Tsay

Department of Information Management
National Taiwan University

Introduction

- 🌐 Logic concerns two concepts:
 - ☀️ **truth** (in a specific or general context)
 - ☀️ **provability** (of truth from assumed truth)
- 🌐 **Formal (symbolic) logic** approaches logic by rules for manipulating symbols:
 - ☀️ **syntax** rules: for writing statements or formulae.
(There are also semantic rules determining whether a statement is true or false in a context or mathematical structure.)
 - ☀️ **inference** rules: for obtaining true statements from other true statements.
- 🌐 We shall introduce two main branches of formal logic:
 - ☀️ **propositional logic**
 - ☀️ **first-order logic** (predicate logic/calculus)
- 🌐 The following slides cover **propositional logic**.

Propositions

- 🌐 A *proposition* is a statement that is either *true* or *false* such as the following:
 - ☀️ Leslie is a teacher.
 - ☀️ Leslie is rich.
 - ☀️ Leslie is a pop singer.
- 🌐 Simplest (*atomic*) propositions may be combined to form *compound* propositions:
 - ☀️ Leslie is *not* a teacher.
 - ☀️ *Either* Leslie is not a teacher *or* Leslie is not rich.
 - ☀️ *If* Leslie is a pop singer, *then* Leslie is rich.

Inferences

- 🌐 We are given the following assumptions:
 - ☀️ Leslie is a teacher.
 - ☀️ Either Leslie is not a teacher or Leslie is not rich.
 - ☀️ If Leslie is a pop singer, then Leslie is rich.
- 🌐 We wish to conclude the following:
 - ☀️ Leslie is not a pop singer.
- 🌐 The above process is an example of *inference* (deduction). Is it correct?

Symbolic Propositions

- Propositions are represented by *symbols*, when only their truth values are of concern.
 - P : Leslie is a teacher.
 - Q : Leslie is rich.
 - R : Leslie is a pop singer.
- Compound propositions can then be more succinctly written.
 - not* P : Leslie is not a teacher.
 - not* P *or* *not* Q : Either Leslie is not a teacher or Leslie is not rich.
 - R *implies* Q : If Leslie is a pop singer, then Leslie is rich.

Symbolic Inferences

- 🌐 We are given the following assumptions:
 - ☀️ P (Leslie is a teacher.)
 - ☀️ $\text{not } P \text{ or not } Q$ (Either Leslie is not a teacher or Leslie is not rich.)
 - ☀️ $R \text{ implies } Q$ (If Leslie is a pop singer, then Leslie is rich.)
- 🌐 We wish to conclude the following:
 - ☀️ $\text{not } R$ (Leslie is not a pop singer.)
- 🌐 Correctness of the inference may be checked by asking:
 - ☀️ Is $(P \text{ and } (\text{not } P \text{ or not } Q) \text{ and } (R \text{ implies } Q)) \text{ implies } (\text{not } R)$ a tautology (valid formula)?
 - ☀️ Or, is $(A \text{ and } (\text{not } A \text{ or not } B) \text{ and } (C \text{ implies } B)) \text{ implies } (\text{not } C)$ a tautology (valid formula)?

Propositional Logic: Syntax

Vocabulary:

-  A countable set \mathcal{P} of *proposition symbols* (variables): P, Q, R, \dots (also called *atomic propositions*);
-  *Logical connectives* (operators): $\neg, \wedge, \vee, \rightarrow,$ and \leftrightarrow and sometimes the constant \perp (*false*);
-  Auxiliary symbols: “(”, “)”.

Propositional Formulae:

-  Any $A \in \mathcal{P}$ is a formula (and so is \perp).
-  If A and B are formulae, then so are $\neg A, (A \wedge B), (A \vee B), (A \rightarrow B),$ and $(A \leftrightarrow B)$.

Propositional Logic: Semantics

- The meanings of propositional formulae may be conveniently summarized by the truth table:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

The meaning of \perp is always F (false).

- There is an implicit inductive definition in the table. We shall try to make this precise.

Truth Assignment and Valuation

- 🌐 The semantics of propositional logic assigns a truth function to each propositional formula.
- 🌐 Let $BOOL$ be the set of truth values $\{T, F\}$.
- 🌐 A *truth assignment* (valuation) is a function from \mathcal{P} (the set of proposition symbols) to $BOOL$.
- 🌐 Let $PROPS$ be the set of all propositional formulae.
- 🌐 A truth assignment v may be extended to a *valuation* function \hat{v} from $PROPS$ to $BOOL$ as follows:

Truth Assignment and Valuation (cont.)

$$\hat{v}(\perp) = F$$

$$\hat{v}(P) = v(P) \text{ for all } P \in \mathcal{P}$$

$\hat{v}(P)$ = as defined by the table below, otherwise

$\hat{v}(A)$	$\hat{v}(B)$	$\hat{v}(\neg A)$	$\hat{v}(A \wedge B)$	$\hat{v}(A \vee B)$	$\hat{v}(A \rightarrow B)$	$\hat{v}(A \leftrightarrow B)$
<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>

Truth Assignment and Satisfaction

- 🌐 We say $v \models A$ (v *satisfies* A) if $\hat{v}(A) = T$.
- 🌐 So, the symbol \models denotes a binary relation, called *satisfaction*, between truth assignments and propositional formulae.
- 🌐 $v \not\models A$ (v *falsifies* A) if $\hat{v}(A) = F$.

🌐 Alternatively, the satisfaction relation \models may be defined directly as follows:

$$v \not\models \perp$$

$$v \models P \iff v(P) = T, \text{ for all } P \in \mathcal{P}$$

$$v \models \neg A \iff v \not\models A \text{ (it is *not* the case that } v \models A)$$

$$v \models A \wedge B \iff v \models A \text{ and } v \models B$$

$$v \models A \vee B \iff v \models A \text{ or } v \models B$$

$$v \models A \rightarrow B \iff v \not\models A \text{ or } v \models B$$

$$v \models A \leftrightarrow B \iff (v \models A \text{ and } v \models B) \\ \text{or } (v \not\models A \text{ and } v \not\models B)$$

Object vs. Meta Language

- 🌐 The language that we study is referred to as the *object* language.
- 🌐 The language that we use to study the object language is referred to as the *meta* language.
- 🌐 For example, *not*, *and*, and *or* that we used to define the satisfaction relation \models are part of the meta language.

- 🌐 A proposition A is *satisfiable* if there exists an assignment v such that $v \models A$.
 - ☀️ $v(P) = F, v(Q) = T \models (P \vee Q) \wedge (\neg P \vee \neg Q)$
- 🌐 A proposition is *unsatisfiable* if no assignment satisfies it.
 - ☀️ $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge P$ is unsatisfiable.
- 🌐 The problem of determining whether a given proposition is satisfiable is called the *satisfiability problem*.

Tautology and Validity

- 🌐 A proposition A is a *tautology* if every assignment satisfies A , written as $\models A$.
 - ☀️ $\models A \vee \neg A$
 - ☀️ $\models (A \wedge B) \rightarrow (A \vee B)$
- 🌐 The problem of determining whether a given proposition is a tautology is called the *tautology problem*.
- 🌐 A proposition is also said to be *valid* if it is a tautology.
- 🌐 So, the problem of determining whether a given proposition is valid (a tautology) is also called the *validity problem*.

Note: The notion of a tautology is restricted to propositional logic. In first-order logic, we also speak of valid formulae.

Validity vs. Satisfiability

Theorem

A proposition A is valid (a tautology) if and only if $\neg A$ is unsatisfiable.

So, there are two ways of proving that a proposition A is a tautology:

- 🌐 A is satisfied by every truth assignment (or A cannot be falsified by any truth assignment).
- 🌐 $\neg A$ is unsatisfiable.

Semantic Entailment

- Consider two sets of propositions Γ and Δ .
- We say that $v \models \Gamma$ (v satisfies Γ) if $v \models B$ for every $B \in \Gamma$; analogously for Δ .
- We say that Δ is a *semantic consequence* of Γ if every assignment that satisfies Γ also satisfies Δ , written as $\Gamma \models \Delta$.
 - $A, A \rightarrow B \models A, B$
 - $A \rightarrow B, \neg B \models \neg A$
- We also say that Γ *semantically entails* Δ when $\Gamma \models \Delta$.

Relating the Logical Connectives

Lemma

$$\models (A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A))$$

$$\models (A \rightarrow B) \leftrightarrow (\neg A \vee B)$$

$$\models (A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$$

$$\models \perp \leftrightarrow (A \wedge \neg A)$$

Note: These equivalences imply that some connectives could be dispensed with. We normally want a smaller set of connectives when analyzing properties of the logic and a larger set when actually using the logic.

Normal Forms

- 🌐 A *literal* is an atomic proposition or its negation.
- 🌐 A propositional formula is in **Conjunctive Normal Form (CNF)** if it is a conjunction of disjunctions of literals.
 - ☀ $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q) \wedge P$
 - ☀ $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
- 🌐 A propositional formula is in **Disjunctive Normal Form (DNF)** if it is a disjunction of conjunctions of literals.
 - ☀ $(P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q) \vee P$
 - ☀ $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R)$
- 🌐 A propositional formula is in **Negation Normal Form (NNF)** if negations occur only in literals.
 - ☀ CNF or DNF is also NNF (but not vice versa).
 - ☀ $(P \wedge \neg Q) \wedge (P \vee (Q \wedge \neg R))$ in NNF, but not CNF or DNF.

Sequents

- 🌐 A (**propositional**) *sequent* is an expression of the form $\Gamma \vdash \Delta$, where $\Gamma = A_1, A_2, \dots, A_m$ and $\Delta = B_1, B_2, \dots, B_n$ are finite (possibly empty) sequences of (**propositional**) formulae.
- 🌐 In a sequent $\Gamma \vdash \Delta$, Γ is called the *antecedent* (also *context*) and Δ the *consequent*.

Note: Many authors prefer to write a sequent as $\Gamma \longrightarrow \Delta$ or $\Gamma \Longrightarrow \Delta$, while reserving the symbol \vdash for provability (deducibility) in the proof (deduction) system under consideration.

Sequents (cont.)

- 🌐 A sequent $A_1, A_2, \dots, A_m \vdash B_1, B_2, \dots, B_n$ is **falsifiable** if there exists a valuation v such that

$$v \models (A_1 \wedge A_2 \wedge \dots \wedge A_m) \wedge (\neg B_1 \wedge \neg B_2 \wedge \dots \wedge \neg B_n).$$
 - ☀ $A \vee B \vdash B$ is falsifiable, as

$$v(A) = T, v(B) = F \models (A \vee B) \wedge \neg B.$$
- 🌐 A sequent $A_1, A_2, \dots, A_m \vdash B_1, B_2, \dots, B_n$ is **valid** if, for every valuation v , $v \models A_1 \wedge A_2 \wedge \dots \wedge A_m \rightarrow B_1 \vee B_2 \vee \dots \vee B_n$.
 - ☀ $A \vdash A, B$ is valid.
 - ☀ $A, B \vdash A \wedge B$ is valid.
- 🌐 A sequent is **valid** if and only if it is **not falsifiable**.
- 🌐 In the following, we will use only sequents of this simpler form:

$$A_1, A_2, \dots, A_m \vdash C,$$
 where C is a formula.

Inference Rules

- 🌐 Inference rules allow one to obtain true statements from other true statements.
- 🌐 Below is an inference rule for conjunction.

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

- 🌐 In an inference rule, the upper sequents (above the horizontal line) are called the *premises* and the lower sequent is called the *conclusion*.

- 🌐 A **deduction tree** is a tree where each node is labeled with a sequent such that, for every internal (non-leaf) node,
 - ☀️ the label of the **node** corresponds to the **conclusion** and
 - ☀️ the labels of its **children** correspond to the **premises**of an instance of an inference rule.
- 🌐 A **proof tree** is a deduction tree, each of whose leaves is labeled with an axiom.
- 🌐 The root of a deduction or proof tree is called the **conclusion**.
- 🌐 A sequent is **provable** if there exists a proof tree of which it is the conclusion.

Natural Deduction in the Sequent Form

$$\frac{}{\Gamma, A \vdash A} (Ax)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} (\wedge I)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} (\wedge E_1)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} (\wedge E_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} (\vee I_1)$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} (\vee I_2)$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} (\vee E)$$

Natural Deduction (cont.)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} (\rightarrow I)$$

$$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} (\rightarrow E)$$

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A} (\neg I)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B} (\neg E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} (\neg\neg I)$$

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} (\neg\neg E)$$

These inference rules collectively are called System *ND* (the propositional part).

Soundness and Completeness

Theorem

System ND is *sound*, i.e., if a sequent $\Gamma \vdash C$ is *provable* in ND , then $\Gamma \vdash C$ is *valid*.

Theorem

System ND is *complete*, i.e., if a sequent $\Gamma \vdash C$ is *valid*, then $\Gamma \vdash C$ is *provable* in ND .

Compactness

A set Γ of propositions is **satisfiable** if some valuation satisfies every proposition in Γ . For example, $\{A \vee B, \neg B\}$ is satisfiable.

Theorem

For any (possibly infinite) set Γ of propositions, if every finite non-empty subset of Γ is satisfiable then Γ is satisfiable.

Proof hint: by contradiction and the completeness of *ND*.

Consistency

- A set Γ of propositions is *consistent* if there exists some proposition B such that the sequent $\Gamma \vdash B$ is not provable.
- Otherwise, Γ is *inconsistent*; e.g., $\{A, \neg(A \vee B)\}$ is inconsistent.

Lemma

*For System ND, a set Γ of propositions is **inconsistent** if and only if there is some proposition A such that both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ are provable.*

Theorem

*For System ND, a set Γ of propositions is **satisfiable** if and only if Γ is **consistent**.*