Correctness Proofs with Hoare Logic

In a proof of program correctness with Hoare logic, there are actually two layers of logical reasoning—the first layer deals with Hoare triples using the inference rules of Hoare logic itself and the second one deals with ordinary predicates using the inference rules of some flavor of first-order logic (predicate calculus). For the second layer, we also need axioms, mostly algebraic laws, about the domains of the variables being manipulated by the program. As our focus is on the first layer, all details of the second layer reasoning will be omitted and collectively referred to as "predicate calculus + algebra". Below are three example proofs.

1. A proof of "$\{x > 0\}\ x := x - 1\ \{x \geq 0\}$":

$$\dfrac{\dfrac{\text{pred. calculus + algebra}}{x > 0 \rightarrow (x-1) \geq 0} \qquad \dfrac{}{\{(x-1) \geq 0\ (\equiv (x \geq 0)[x-1/x])\}\ x := x - 1\ \{x \geq 0\}}\ (\text{Assign.})}{\{x > 0\}\ x := x - 1\ \{x \geq 0\}}\ (\text{S. Pre.})$$

2. A proof of "$\{x \geq 0\}$ **if** $x > 0$ **then** $x := x - 1$ **fi** $\{x \geq 0\}$":

$$\dfrac{\dfrac{\dfrac{\text{pred. calculus + algebra}}{x \geq 0 \wedge x > 0 \rightarrow x > 0} \qquad \dfrac{\text{same as in Example 1}}{\{x > 0\}\ x := x - 1\ \{x \geq 0\}}}{\{x \geq 0 \wedge x > 0\}\ x := x - 1\ \{x \geq 0\}}\ (\text{S. Pre.}) \qquad \dfrac{\text{pred. calculus + algebra}}{x \geq 0 \wedge \neg(x > 0) \rightarrow x \geq 0}}{\{x \geq 0\}\ \textbf{if}\ x > 0\ \textbf{then}\ x := x - 1\ \textbf{fi}\ \{x \geq 0\}}\ (\text{If} - \text{Then})$$

3. A proof of "$\{x \geq 0\}$ **while** $x > 0$ **do** $x := x - 1$ **od** $\{x = 0\}$":

$$\dfrac{\dfrac{\dfrac{\text{same as the left subtree in Example 2}}{\{x \geq 0 \wedge x > 0\}\ x := x - 1\ \{x \geq 0\}}}{\{x \geq 0\}\ \textbf{while}\ x > 0\ \textbf{do}\ x := x - 1\ \textbf{od}\ \{x \geq 0 \wedge \neg(x > 0)\}}\ (\text{while}) \qquad \dfrac{\text{pred. calculus + algebra}}{x \geq 0 \wedge \neg(x > 0) \rightarrow x = 0}}{\{x \geq 0\}\ \textbf{while}\ x > 0\ \textbf{do}\ x := x - 1\ \textbf{od}\ \{x = 0\}}\ (\text{W. Post.})$$