Concurrency:
Hoare Logic (III)

(Based on [Apt and Olderog 1997; Lamport 1980; Owicki and Gries 1976])

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Sequential vs. Concurrent Programs

- Sequential programs (components) with the same input/output behavior may behave differently when executed in parallel with some other component.
- Consider two program components:

$$S_1 \triangleq x := x + 2 \text{ and } S_1' \triangleq x := x + 1; x := x + 1.$$

  Both increment x by 2.
- When executed in parallel with

$$S_2 \triangleq x := 0,$$

  S_1 and S_1' behave differently.

Sequential vs. Concurrent Programs (cont.)

Indeed,

$$\{true\}\ [S_1 \| S_2]\ \{x = 0 \vee x = 2\}$$

i.e.,

$$\{true\}\ [x := x + 2 \| x := 0]\ \{x = 0 \vee x = 2\}$$

but

$$\{true\}\ [S_1' \| S_2]\ \{x = 0 \vee x = 1 \vee x = 2\}$$

i.e.,

$$\{true\}\ [x := x + 1; x := x + 1 \| x := 0]\ \{x = 0 \vee x = 1 \vee x = 2\}.$$

Atomicity and Interleaving

- An action A (a statement or boolean expression) of a component is called *atomic* if during its execution no other components may change the variables of A.
- The computation of each component can be thought of as a sequence of executions of atomic actions.
- An atomic action is said to be *enabled* if its containing component is ready to execute it.
- Atomic actions enabled in different components are executed in an arbitrary sequential order; this is called the *interleaving* model.

Extending Hoare Logic

The best-known attempt at generalizing Hoare Logic to concurrent programs is:

> S. Owicki and D. Gries. An axiomatic proof technique for parallel programs. *Acta Informatica*, 6:319-340, 1976.

- Proof outlines (for terminating programs)
- Interference freedom
- Auxiliary variables

Proof Outlines

Let S^* stand for a program S annotated with assertions. A proof outline (for partial correctness) is defined by the following formation rules.

$$\frac{}{\{P\}\ \mathbf{skip}\ \{P\}} \qquad \text{(Skip)}$$

$$\frac{}{\{Q[E/x]\}\ x := E\ \{Q\}} \qquad \text{(Assignment)}$$

$$\frac{\{P\}\ S_1^*\ \{R\} \qquad \{R\}\ S_2^*\ \{Q\}}{\{P\}\ S_1^*; \{R\}\ S_2^*\ \{Q\}} \qquad \text{(Sequence)}$$

$$\frac{\{P \wedge B\}\ S_1^*\ \{Q\} \qquad \{P \wedge \neg B\}\ S_2^*\ \{Q\}}{\{P\}\ \mathbf{if}\ B\ \mathbf{then}\ \{P \wedge B\}\ S_1^*\ \{Q\}\ \mathbf{else}\ \{P \wedge \neg B\}\ S_2^*\ \{Q\}\ \mathbf{fi}\ \{Q\}} \qquad \text{(Conditional)}$$

Proof Outlines (cont.)

$$\frac{\{P \wedge B\}\ S^*\ \{P\}}{\{\mathbf{inv} : P\}\ \mathbf{while}\ B\ \mathbf{do}\ \{P \wedge B\}\ S^*\ \{P\}\ \mathbf{od}\ \{P \wedge \neg B\}} \quad \textbf{(while)}$$

$$\frac{P \rightarrow P' \qquad \{P'\}\ S^*\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ \{P'\}\ S^*\ \{Q'\}\ \{Q\}} \quad \text{(Consequence)}$$

$$\frac{\{P\}\ S^*\ \{Q\}}{\{P\}\ S^{**}\ \{Q\}} \quad \text{(Omission)}$$

where S^{**} is obtained from S^* by omitting some of the intermediate assertions not labeled by **inv**.

A proof outline $\{P\}\ S^*\ \{Q\}$ is said to be *standard* if every subprogram T of S is preceded by exactly one assertion, called $pre(T)$, and there are no other assertions.

Atomic Regions

- We enclose multiple statements in a pair of "$\langle$" and "$\rangle$" to form *atomic regions* such as $\langle S_1; S_2 \rangle$, indicating that the enclosed statements are to be executed atomically.
- Proof rule:

$$\frac{\{P\}\ S\ \{Q\}}{\{P\}\ \langle S \rangle\ \{Q\}} \qquad \text{(Atomic Region)}$$

- Proof outline formation:

$$\frac{\{P\}\ S^*\ \{Q\}}{\{P\}\ \langle S^* \rangle\ \{Q\}} \qquad \text{(Atomic Region)}$$

- A proof outline with atomic regions is standard if every normal subprogram is preceded by exactly one assertion (and there are no other assertions).

Interference Freedom

- A standard proof outline $\{p_i\}\ S_i^*\ \{q_i\}$ *does not interfere* with another proof outline $\{p_j\}\ S_j^*\ \{q_j\}$ if the following holds:

  For every normal assignment or atomic region R in S_i and every assertion r in $\{p_j\}\ S_j^*\ \{q_j\}$,

$$\{r \wedge pre(R)\}\ R\ \{r\}.$$

- Given a parallel program $[S_1 \| \cdots \| S_n]$, the standard proof outlines $\{p_i\}\ S_i^*\ \{q_i\}$, $1 \leq i \leq n$, are said to be *interference free* if none of the proof outlines interferes with any other.

Interference Freedom (cont.)

- Proof rule:

$$\frac{\{p_i\}\ S_i^*\ \{q_i\},\ 1 \le i \le n, \text{ are standard and interference free}}{\{\bigwedge_{i=1}^{n} p_i\}\ [S_1 \| \cdots \| S_n]\ \{\bigwedge_{i=1}^{n} q_i\}}$$

An Example

$$\{x = 0\} \quad \{true\}$$
$$x := x + 2 \quad x := 0$$
$$\{x = 2\} \quad \{x = 0\}$$

are not interference free.

$$\{x = 0\} \quad \{true\}$$
$$x := x + 2 \quad x := 0$$
$$\{x = 0 \lor x = 2\} \quad \{x = 0 \lor x = 2\}$$

are interference free and yield

$$\{x = 0\}\ [x := x + 2 \| x := 0]\ \{x = 0 \lor x = 2\}.$$

An Example (cont.)

- Can we prove the following stronger claim?

$$\{true\}\ [x := x + 2 \| x := 0]\ \{x = 0 \vee x = 2\}$$

- This is not possible if we rely only on the proof rules introduced so far.
- It is easy to see that we must prove, for some q_1 and q_2,

$$\{true\}\ [x := x + 2]\ \{q_1\} \text{ and } \{true\}\ [x := 0]\ \{q_2\}.$$

From $\{true\}\ [x := x + 2]\ \{q_1\}$, q_1 equals $true$ and hence q_2 along must imply $(x = 0 \vee x = 2)$.
    - From $\{true\}\ [x := 0]\ \{q_2\}$, $q_2[0/x]$ holds.
    - From $\{true \wedge q_2\}\ [x := x + 2]\ \{q_2\}$, $q_2 \rightarrow q_2[x + 2/x]$ holds.
    - By induction, q_2 holds for all even x's, a contradiction.

Auxiliary Variables

- A variable z in a program is called auxiliary if it only appears in assignments of the form $z := t$.
- Rule for auxiliary variables

$$\frac{\{p\}\ S\ \{q\}}{\{p\}\ S_0\ \{q\}} \qquad \text{(Auxiliary Variables)}$$

where S_0 is obtained from S by deleting some assignments with an auxiliary variable that does not occur free in q.

An Example (cont.)

$$\{\neg done\} \qquad \{true\}$$
$$\langle x := x + 2; done := true\rangle \qquad x := 0$$
$$\{true\} \qquad \{(x = 0 \lor x = 2) \land (\neg done \rightarrow x = 0)\}.$$

are interference free and yield

$$\{\neg done\}$$
$$[\langle x := x + 2; done := true\rangle \| x := 0]$$
$$\{(x = 0 \lor x = 2) \land (\neg done \rightarrow x = 0)\}$$

The conjunct $(\neg done \rightarrow x = 0)$ can now be dropped (for our purpose).

An Example (cont.)

$$\{true\}$$
$$done := false;$$
$$\{\neg done\}$$
$$[\langle x := x + 2; done := true\rangle \| x := 0]$$
$$\{x = 0 \lor x = 2\}$$

from which we infer

$$\{true\}$$
$$[x := x + 2 \| x := 0]$$
$$\{x = 0 \lor x = 2\}.$$

The await Statement

- Syntax:

$$\textbf{await}\ B\ \textbf{then}\ S\ \textbf{end}$$

  The special case "$\textbf{await}\ B\ \textbf{then}\ skip\ \textbf{end}$" is simply written as "$\textbf{await}\ B$".

- Semantics:
  If B evaluates to $true$, S is executed as an atomic region and the component then proceeds to the next action. If B evaluates to $false$, the component is *blocked* and continues to be blocked unless B becomes $true$ later (because of the executions of other components).

The await Statement (cont.)

- Proof rule:

$$\frac{\{P \wedge B\}\ S\ \{Q\}}{\{P\}\ \textbf{await}\ B\ \textbf{then}\ S\ \textbf{end}\ \{Q\}} \qquad \textbf{(await)}$$

- Proof outline formation:

$$\frac{\{P \wedge B\}\ S^*\ \{Q\}}{\{P\}\ \textbf{await}\ B\ \textbf{then}\ \{P \wedge B\}\ S^*\ \{Q\}\ \textbf{end}\ \{Q\}} \qquad \textbf{(await)}$$

- For a proof outline to be standard, assertions within an **await** statement must be removed.

An Example with await

$$\begin{array}{ll} \ldots & \ldots \\ Q[0] := true; & Q[1] := true; \\ \textbf{await } \neg Q[1]; & \textbf{await } \neg Q[0]; \\ \text{/* critical section */} & \text{/* critical section */} \\ Q[0] := false; & Q[1] := false; \\ \ldots & \ldots \end{array}$$

Note 1: This is the “first half” of Peterson’s algorithm for two-process mutual exclusion.

Note 2: $Q[0]$ and $Q[1]$ are $false$ initially.

An Example with await (cont.)

$\{\neg Q[0]\}$	$\{\neg Q[1]\}$
$Q[0] := \mathit{true}$;	$Q[1] := \mathit{true}$;
$\{Q[0]\}$	$\{Q[1]\}$
await $\neg Q[1]$;	**await** $\neg Q[0]$;
$\{Q[0]\}$	$\{Q[1]\}$
$Q[0] := \mathit{false}$;	$Q[1] := \mathit{false}$;
$\{\neg Q[0]\}$	$\{\neg Q[1]\}$

Note: interference free, but not very useful . . . .
We should look for assertions at the two critical sections such that their conjunction results in a contradiction.

An Example with await (cont.)

$\{\neg Q[0]\}$
$Q[0] := true;$
$\{Q[0]\}$
$\textbf{await}\ \neg Q[1];$
$\{Q[0] \wedge \neg Q[1]\}$
$Q[0] := false;$
$\{\neg Q[0]\}$

$\{\neg Q[1]\}$
$Q[1] := true;$
$\{Q[1]\}$
$\textbf{await}\ \neg Q[0];$
$\{Q[1] \wedge \neg Q[0]\}$
$Q[1] := false;$
$\{\neg Q[1]\}$

Note: looks useful, but not interference free . . . .

An Example with await (cont.)

$\{\neg Q[0]\}$
$\langle Q[0], X[0] := true, true; \rangle$
$\{Q[0] \wedge X[0]\}$
$\langle \textbf{await}\ \neg Q[1]; X[0] := false; \rangle$
$\{Q[0] \wedge \neg X[0] \wedge (\neg Q[1] \vee X[1])\}$
$Q[0] := false;$
$\{\neg Q[0]\}$

$\{\neg Q[1]\}$
$\langle Q[1], X[1] := true, true; \rangle$
$\{Q[1] \wedge X[1]\}$
$\langle \textbf{await}\ \neg Q[0]; X[1] := false; \rangle$
$\{Q[1] \wedge \neg X[1] \wedge (\neg Q[0] \vee X[0])\}$
$Q[1] := false;$
$\{\neg Q[1]\}$

Note 1: "$\langle \textbf{await}\ \neg Q[0]; X[1] := false; \rangle$" is a shorter form for "$\textbf{await}\ \neg Q[0]\ \textbf{then}\ X[1] := false\ \textbf{end}$".

Note 2: conjoining the two assertions at the two critical sections gives the needed contradiction.

Lamport's 'Hoare Logic'

In this probably forgotten paper, Lamport proposed a new interpretation to pre and post-conditions:

L. Lamport. The 'Hoare Logic' of concurrent programs. *Acta Informatica*, 14:21-37, 1980.

- Notation: $\{P\}\ S\ \{Q\}$
  Meaning: If execution starts anywhere in S with P true, then executing S (1) will leave P true while control is in S and (2) if terminating, will make Q true.
- The usual Hoare triple would be expressed as $\{P\}\ \langle S \rangle\ \{Q\}$, where $\langle \cdot \rangle$ indicates atomic execution.

Lamport's 'Hoare Logic' (cont.)

- Rule of consequence (can't strengthen the pre-condition):

$$\frac{\{P\}\ S\ \{Q'\},\ Q' \to Q}{\{P\}\ S\ \{Q\}}$$

- Rules of Conjunction and Disjunction:

$$\frac{\{P\}\ S\ \{Q\},\ \{P'\}\ S\ \{Q'\}}{\{P \wedge P'\}\ S\ \{Q \wedge Q'\}} \qquad \frac{\{P\}\ S\ \{Q\},\ \{P'\}\ S\ \{Q'\}}{\{P \vee P'\}\ S\ \{Q \vee Q'\}}$$

Lamport's 'Hoare Logic' (cont.)

- Rule of Sequential Composition:

$$\frac{\{P\}\ S\ \{Q\},\ \{R\}\ T\ \{U\},\ Q \wedge at(T) \rightarrow R}{\{(in(S) \rightarrow P) \wedge (in(T) \rightarrow R)\}\ S;T\ \{U\}}$$

- Rule of Parallel Composition:

$$\frac{\{P\}\ S_i\ \{P\},\ 1 \leq i \leq n}{\{P\}\ \mathbf{cobegin}\ \mathop{\|}_{i=1}^{n}\ S_i\ \mathbf{coend}\ \{P\}}$$