Suggested Solutions for Homework Assignment #2

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\forall, \exists\}$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. Prove, using Gentzen's System LK, the validity of the following sequents:

   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall xP(x) \rightarrow \forall xQ(x)$ (10 points)

   (b) $\vdash \exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)$ (10 points)

   (c) $\exists xA(x) \rightarrow B \vdash \forall x(A(x) \rightarrow B)$, assuming x does not occur free in B. (10 points)

   Solution.

   (a) Assume w does not occur free both in $P(x)$ and in $Q(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(w) \vdash P(w)}\,\mathit{Axiom} \qquad \dfrac{}{Q(w), P(w) \vdash Q(w)}\,\mathit{Axiom}}{P(x)[w/x] \rightarrow Q(x)[w/x], P(x)[w/x] \vdash Q(w)}\,\rightarrow: L}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash Q(x)[w/x]}\,\forall: L}{\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x)}\,\forall: R}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall xP(x) \rightarrow \forall xQ(x)}\,\rightarrow: R$$

   (b) Assume both w and z do not occur free in $P(x,y)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(z,y)[w/y]\{= P(z,w)\} \vdash P(z,w)}\,\mathit{Axiom}}{\forall yP(z,y) \vdash P(x,w)[z/x]}\,\forall: L}{(\forall yP(x,y))[z/x] \vdash \exists xP(x,w)}\,\exists: R}{\exists x\forall yP(x,y) \vdash (\exists xP(x,y))[w/y]}\,\exists: L}{\exists x\forall yP(x,y) \vdash \forall y\exists xP(x,y)}\,\forall: R}{\vdash \exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)}\,\rightarrow: R$$

   (c) Assume w does not occur free both in $A(x)$ and in B.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A(w) \vdash A(w)}\,\mathit{Axiom}}{A(w) \vdash \exists xA(x)}\,\exists: R \qquad \dfrac{}{A(w), B \vdash B}\,\mathit{Axiom}}{\exists xA(x) \rightarrow B, A(w) \vdash B}\,\rightarrow: L}{\exists xA(x) \rightarrow B \vdash A(w) \rightarrow B}\,\rightarrow: R}{\exists xA(x) \rightarrow B \vdash \forall x(A(x) \rightarrow B)}\,\forall: R$$

   □

2. Prove, using *Natural Deduction*, the validity of the following sequents:

   (a) $\forall x(P(x) \rightarrow Q(x)) \vdash \forall xP(x) \rightarrow \forall xQ(x)$ (10 points)

   (b) $\vdash \exists x\forall yP(x,y) \rightarrow \forall y\exists xP(x,y)$ (10 points)

(c) $\forall x(A(x) \to B) \vdash \exists x A(x) \to B$, assuming x does not occur free in B. (10 points)

Solution.

(a) Assume w does not occur free both in $P(x)$ and in $Q(x)$.

$$\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x P(x)}\,Hyp}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash P(w)}\,\forall E}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash Q(w)}\,\to E}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)}\,\forall I}{\forall x(P(x) \to Q(x)) \vdash \forall x P(x) \to \forall x Q(x)}\,\to I$$

α :

$$\dfrac{\dfrac{}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x(P(x) \to Q(x))}\,Hyp}{\forall x(P(x) \to Q(x)), \forall x P(x) \vdash P(w) \to Q(w)}\,\forall E$$

(b) Assume both w and z do not occur free in $P(x, y)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x \forall y P(x,y) \vdash \exists x \forall y P(x,y)}\,Hyp \qquad \dfrac{\dfrac{\dfrac{}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash \forall y P(z,y)}\,Hyp}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash P(z,w)}\,\forall E}{\exists x \forall y P(x,y), \forall y P(z,y) \vdash \exists x P(x,w)}\,\exists I}{\exists x \forall y P(x,y) \vdash \exists x P(x,w)}\,\exists E}{\exists x \forall y P(x,y) \vdash \forall y \exists x P(x,y)}\,\forall I}{\vdash \exists x \forall y P(x,y) \to \forall y \exists x P(x,y)}\,\to I$$

(c) Assume w does not occur free both in $A(x)$ and in B.

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists x A(x) \vdash \exists x A(x)}\,Hyp \qquad \alpha}{\forall x(A(x) \to B), \exists x A(x) \vdash B}\,\exists E}{\forall x(A(x) \to B) \vdash \exists x A(x) \to B}\,\to I$$

α :

$$\dfrac{\dfrac{\dfrac{}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash \forall x(A(x) \to B)}\,Hyp}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash A(w) \to B}\,\forall E \qquad \beta}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash B}\,\to E$$

β :

$$\dfrac{}{\forall x(A(x) \to B), \exists x A(x), A(w) \vdash A(w)}\,Hyp$$

□

3. Prove, using *Natural Deduction* for the first-order logic with equality (=), that = is an equivalence relation between terms, i.e., the following are valid sequents, in addition to the obvious "$\vdash t = t$" (Reflexivity), which follows from the =-Introduction rule.

(a) $t_2 = t_1 \vdash t_1 = t_2$ (Symmetry) (10 points)

(b) $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$ (Transitivity) (10 points)

Solution.

(a)

$$\dfrac{\dfrac{}{t_2 = t_1 \vdash t_2 = t_1}\,Hyp \qquad \dfrac{}{t_2 = t_1 \vdash t_2 = t_2}\,{=}I}{t_2 = t_1 \vdash t_1 = t_2}\,{=}E$$

(b)

$$\dfrac{\dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_2 = t_3}\,Hyp \qquad \dfrac{}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_2}\,Hyp}{t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,{=}E$$

□

4. Taking the preceding valid sequents as axioms, prove using *Natural Deduction* the following derived rules for equality.

(a) $\dfrac{\Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}$ (= *Symmetry*) (10 points)

(b) $\dfrac{\Gamma \vdash t_1 = t_2 \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}$ (= *Transitivity*) (10 points)

Solution.

(a)

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_2 = t_1 \vdash t_1 = t_2}\,Axiom(Symmetry)}{\Gamma \vdash t_2 = t_1 \to t_1 = t_2}\,{\to}I \qquad \Gamma \vdash t_2 = t_1}{\Gamma \vdash t_1 = t_2}\,{\to}E$$

(b)

$$\dfrac{\dfrac{\alpha \qquad \Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_3 \to t_1 = t_3}\,{\to}E \qquad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}\,{\to}E$$

α :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3}\,Axiom(Transitivity)}{\Gamma, t_1 = t_2 \vdash t_2 = t_3 \to t_1 = t_3}\,{\to}I}{\Gamma \vdash t_1 = t_2 \to (t_2 = t_3 \to t_1 = t_3)}\,{\to}I$$

□