The Sequent Calculus: System LK

- Axioms

$$\frac{}{A \vdash A}\ (\mathit{Axiom})$$

For convenience, you may extend this to the following:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ (\mathit{Axiom})$$

- Logical Rules (I)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\ (\land : \mathit{Left}_1) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta}\ (\land : \mathit{Right})$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\ (\land : \mathit{Left}_2)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta}\ (\lor : \mathit{Left}) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \lor B, \Delta}\ (\lor : \mathit{Right}_1)$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \lor B, \Delta}\ (\lor : \mathit{Right}_2)$$

For convenience, you may take the following alternatives:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\ (\land : \mathit{Left}) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta}\ (\land : \mathit{Right})$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta}\ (\lor : \mathit{Left}) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \lor B, \Delta}\ (\lor : \mathit{Right})$$

- Logical Rules (II)

$$\frac{\Gamma \vdash A, \Delta_1 \qquad \Gamma, B \vdash \Delta_2}{\Gamma, A \rightarrow B \vdash \Delta_1, \Delta_2}\ (\rightarrow : \mathit{Left}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\ (\rightarrow : \mathit{Right})$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg : \mathit{Left}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg : \mathit{Right})$$

- Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\mathit{Weakening} : \mathit{Left}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,(\mathit{Weakening} : \mathit{Right})$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\mathit{Contraction} : \mathit{Left}) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,(\mathit{Contraction} : \mathit{Right})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\,(\mathit{Exchange} : \mathit{Left}) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\,(\mathit{Exchange} : \mathit{Right})$$

For convenience, you may treat Γ, Δ, etc. as *sets*, A, B as $\{A\}$, $\{B\}$, and the comma (in "Γ, A" etc.) as set union, so that you can do without these structural rules; use this alternative with the extended notion of an axiom.

- The Cut Rule

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\mathit{Cut})$$

- Quantifier Rules

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\,(\forall : \mathit{Left}) \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\,(\forall : \mathit{Right})$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\,(\exists : \mathit{Left}) \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\,(\exists : \mathit{Right})$$

In these quantifier rules, we assume that all substitutions are admissible, y is not free in A, and y does not occur free in the lower sequent.

Axioms for Equality (an extension for languages with $=$):

Let $t, s_1, \cdots, s_n, t_1, \cdots, t_n$ be arbitrary terms.

$$\frac{}{\Gamma \vdash t = t, \Delta}$$

For every n-ary function f,

$$\frac{}{\Gamma, s_1 = t_1, \cdots, s_n = t_n \vdash f(s_1, \cdots, s_n) = f(t_1, \cdots, t_n), \Delta}$$

For every n-ary predicate P (including $=$),

$$\frac{}{\Gamma, s_1 = t_1, \cdots, s_n = t_n, P(s_1, \cdots, s_n) \vdash P(t_1, \cdots, t_n), \Delta}$$