Proofs in the Sequent Calculus (System LK)

- Below is a proof of $p \to q \vdash \neg p \lor q$. Note that the converse, i.e., $\neg p \lor q \vdash p \to q$, is also provable, affirming the equivalence of $p \to q$ and $\neg p \lor q$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathit{Axiom})}{\vdash \neg p, p}\,(\neg : \mathit{Right})}{\vdash \neg p \lor q, p}\,(\lor : \mathit{Right}_1)}{\vdash p, \neg p \lor q}\,(\mathit{Exchange} : \mathit{Right}) \qquad \dfrac{\dfrac{}{q \vdash q}\,(\mathit{Axiom})}{q \vdash \neg p \lor q}\,(\lor : \mathit{Right}_2)}{p \to q \vdash \neg p \lor q, \neg p \lor q}\,(\to : \mathit{Left})}{p \to q \vdash \neg p \lor q}\,(\mathit{Contraction} : \mathit{Right})$$

- If we treat Γ and Δ in a sequent $\Gamma \vdash \Delta$ as *sets* of formulae, then we can do without the structural rules. The preceding proof can be simplified as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(\mathit{Axiom})}{\vdash p, \neg p}\,(\neg : \mathit{Right})}{\vdash p, \neg p \lor q}\,(\lor : \mathit{Right}_1) \qquad \dfrac{\dfrac{}{q \vdash q}\,(\mathit{Axiom})}{q \vdash \neg p \lor q}\,(\lor : \mathit{Right}_2)}{p \to q \vdash \neg p \lor q}\,(\to : \mathit{Left})$$

- Below is a proof of $\vdash \exists x(A \land B) \to \exists x A \land \exists x B$. Its converse does not necessarily hold and hence is not provable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[y/x] \vdash A[y/x]}\,(\mathit{Axiom})}{A[y/x] \land B[y/x] \vdash A[y/x]\ \{y \text{ as } t \text{ in A[t/x]}\}}\,(\land : \mathit{Left}_1)}{A[y/x] \land B[y/x]\ \{=\ (A \land B)[y/x]\} \vdash \exists x A}\,(\exists : \mathit{Right})}{\exists x(A \land B) \vdash \exists x A}\,(\exists : \mathit{Left}) \qquad \dfrac{\text{analogous to the left}}{\exists x(A \land B) \vdash \exists x B}}{\exists x(A \land B) \vdash \exists x A \land \exists x B}\,(\land : \mathit{Right})}{\vdash \exists x(A \land B) \to \exists x A \land \exists x B}\,(\to : \mathit{Right})$$

- A proof of $\forall x(x \geq 0) \vdash \forall x(\forall y(x + y \geq 0))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(w + z \geq 0)\ \{=\ (x \geq 0)[w + z/x]\} \vdash (w + z \geq 0)}\,(\mathit{Axiom})}{\forall x(x \geq 0) \vdash (w + z \geq 0)\ \{=\ (w + y \geq 0)[z/y]\}}\,(\forall : \mathit{Left})}{\forall x(x \geq 0) \vdash \forall y(w + y \geq 0)\ \{=\ (\forall y(x + y \geq 0))[w/x]\}}\,(\forall : \mathit{Right})}{\forall x(x \geq 0) \vdash \forall x(\forall y(x + y \geq 0))}\,(\forall : \mathit{Right})$$