The B-Method

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Main References

1. J.-R. Abrial, The B-Book
   Cambridge University Press, 1996
2. J.B. Wordsworth, Software Engineering with B
   Addison-Wesley, 1996

Agenda

IM NTU

- Introduction
- Specification
- Refinement
- Implementation

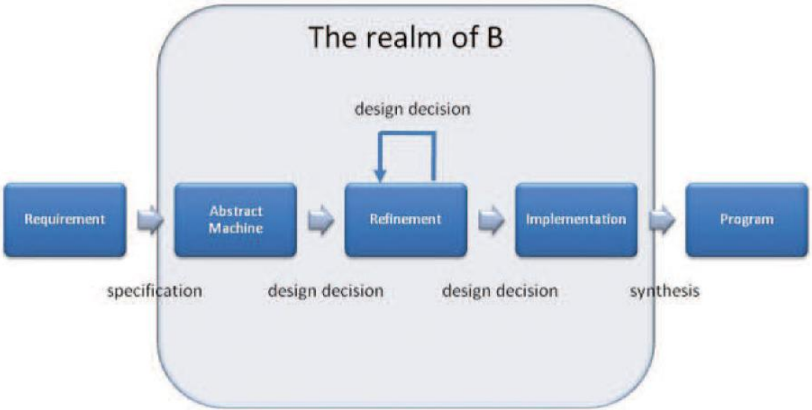
What is the B

- B is a method for specifying, designing, and coding software systems
- B uses a simple "pseudo" programming language to model requirements, to specify interfaces, and to provide implementations and intermediate designs
- The language is known as AMN (Abstract Machine Notation)

Abstract Machine Notation

- In B method, we use abstract machine represents the software model.
- We care about the essential properties of the internal data, not the implementation details.
- Tells what it does, rather than how it does.
- Abstract Machine Notation allows specifications to be statically type checked, dynamically validated, and mathematically verified by proof to ensure the correctness of the design process.

the Global View



Theoretical Basis

- First-order logic
- Set theory
- Integer arithmetics
- Generalized substitutions

Generalized Substitutions

- The mean to describe state changes
- A generalized substitution acts as a predicate transformer.

  The substitution $[V := V + 1]$ substitutes all occurrences of V with the expression V+1, e.g.

  $[V := V + 1]V > 0 \Longleftrightarrow V + 1 > 0$

Generalized Substitutions (cont'd)

- The simple substitution $[V := E]$ has the usual meaning
- Generalized substitutions are expressed by means of simple substitutions and logic operations
- The application of a generalized substitution S to a predicate P yields a new predicate denoted $[S]P$
- If predicate P characterizes a set of states, substitution S represents a state transformation, then $[S]P$ is the predicate characterizing the states such that, when S is applied, the resulting states are in P
- $[S]P$ can be read as "S establishes P"

Generalized Substitutions Forms

- **simple substitution**
  $[x := E]R \Leftrightarrow$ replacing all free occurrences of x in R by E.
- **multiple substitution**
  $[x_1, .., x_n := E_1, .., E_n]R \Leftrightarrow$ simultaneously replacing all free occurrences of $x_1, .., x_n$ in R by $E_1, .., E_n$ respectively.
- **no operation**
  $[skip]R \Leftrightarrow$ R

Generalized Substitutions Forms (cont'd)

- **substitution with precondition**
  $[P|S]R \Leftrightarrow P \wedge [S]R$
- **guarded substitution**
  $[P \Longrightarrow S]R \Leftrightarrow P \Rightarrow [S]R$
- **bounded choice**
  $[S1[]S2]R \Leftrightarrow [S1]R \wedge [S2]R$.
- **unbounded choice**
  $[@z \cdot S]R \Leftrightarrow \forall z \cdot [S]R$, where z is not free in R.

Multiple Substitution

$[V, W := E, F]P \Longleftrightarrow [tmp := F][V := E][W := tmp]P$
where *tmp* is a fresh variable

$[V, W := W, V]V > W$
$\Longleftrightarrow [tmp := V][V := W][W := tmp]V > W$
$\Longleftrightarrow [tmp := V][V := W]V > tmp$
$\Longleftrightarrow [tmp := V]W > tmp$
$\Longleftrightarrow W > V$

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Formal Specification

- Specification(functional) describes an abstract program
- If the initial state satisfies the precondition, then it changes only variables and terminates in a final state satisfying the postcondition
- For validation of the user requirements
  "Is the specified system what the customer wants?"
- To support verification of the code
  "Is the code a correct implementation of the specification?"

Abstract Machines in B

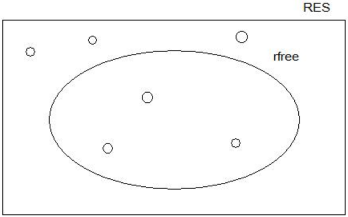
- The basic module for specification in B is the abstract machine
- The abstract machine is a module that consists of
  - a static part defining the state
    - variables (the local state of the abstract machine)
    - invariant (the static laws of the system, properties and requirements)
  - a dynamic part modifying this state
    - operations (modify the state according to the invariant, model services)
- Valid states need to be explicitly specified with an invariant predicate

Basic Structure of the Abstract Machine

MACHINE
  Name(Parameters)
VARIABLES
  list of variables
INVARIANT
  invariant predicate
INITIALISATION
  initialization substitution
OPERATIONS
  $outputs \leftarrow name(inputs) = substitution$
END

(some clauses provided by the B notation for specification are omitted)

Abstract Machine Example



MACHINE
RMan (RES)
VARIABLES
rfree
INVARIANT
$rfree \subseteq RES$
INITIALISATION
$rfree := \phi$

Abstract Machine Example (cont'd)

OPERATIONS

alloc $(rr) =$

PRE $rr \in rfree$

THEN

$rfree := rfree - \{rr\}$

END;

free $(rr) =$

PRE

$rr \in RES \wedge rr \notin rfree$

THEN

$rfree := rfree \cup \{rr\}$

END;

END

Verification of the Specification

- The abstract machine shall initiate in a valid state:
  The initialization shall establish the invariant
- The operations of the abstract machine shall not take it into an invalid state, assuming that their pre-conditions are respected:
  The operations shall preserve the invariant
- Need to generate and discharge proof obligations

Proof Obligations for the Specification's Initialization

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The initialization predicate shall establish the invariant:

$$[G]Inv$$

where G is the initialization substitution, and Inv is the invariant predicate
In the case of the *Resource manager* machine:

$$[rfree := \phi]\ rfree \subseteq RES$$
$$\iff (\phi \subseteq RES)$$
$$\iff \top$$

Proof Obligations for the Specification's Operations

Operations shall preserve the invariant, assuming their precondition is satisfied:

$$Inv \wedge P \Rightarrow [S]Inv$$

where

- Inv is the invariant predicate
- P is the pre-condition of the operation
- S is the substitution of the operation

Example of the Specification's Operation

In the case of the *Resource manager* machine:
INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
PRE $rr \in rfree$
THEN
$rfree := rfree - \{rr\}$
END

$(\; Inv \wedge P \;)$
$\Rightarrow [S]Inv$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:
INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
  PRE $rr \in rfree$
  THEN
    $rfree := rfree - \{rr\}$
  END

$(rfree \subseteq RES \land P)$
$\Rightarrow [S]Inv$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:

INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
  PRE $rr \in rfree$
  THEN
    $rfree := rfree - \{rr\}$
  END

$(\ rfree \subseteq RES \wedge rr \in rfree\ )$
$\Rightarrow [S]Inv$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:
INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
PRE $rr \in rfree$
THEN
$rfree := rfree - \{rr\}$
END

$(\; rfree \subseteq RES \wedge rr \in rfree \;)$
$\Rightarrow [rfree := rfree - \{rr\}]Inv$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:
INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
  PRE $rr \in rfree$
  THEN
    $rfree := rfree - \{rr\}$
  END

$(\ rfree \subseteq RES \wedge rr \in rfree\ )$
$\Rightarrow [rfree := rfree - \{rr\}]rfree \subseteq RES$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:
INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
PRE $rr \in rfree$
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$(\ rfree \subseteq RES \wedge rr \in rfree \ )$
$\Rightarrow [rfree := rfree - \{rr\}]rfree \subseteq RES$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:

INVARIANT $rfree \subseteq RES$

$alloc(rr) =$
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$(\ rfree \subseteq RES \wedge rr \in rfree \ )$
$\Rightarrow rfree - \{rr\} \subseteq RES$

Example of the Specification's Operation (cont'd)

In the case of the *Resource manager* machine:

INVARIANT $rfree \subseteq RES$

$alloc(rr) =$

PRE $rr \in rfree$

THEN

$rfree := rfree - \{rr\}$

END

$(\; rfree \subseteq RES \wedge rr \in rfree \;)$

$\Rightarrow rfree - \{rr\} \subseteq RES$

$\Rightarrow \top$

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Generalities on Refinement

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Refinement

- Is a step on the path from specification to implementation
- Reflects a design decision
- Is better in the sense that it is more accurate, applies in more situations, or runs more efficiently
- Defines the concrete version of the specification variables in the initialization and operations
- A so-called refinement relation establishes the mapping between the values of concrete variables and abstract variables

the Different Kinds of Refinement

The different kinds of refinement correspond to the different kinds of design decisions:

- Data refinement
  introducing data structures that can be easily programmed
- Operation refinement provides indications of how operations are to be computed by algorithms, and may also:
  - allow more input values
  - restrict or even remove non-determinism

Refinements in B

- A refinement is a module introduced by the keyword:REFINEMENT
- The refined machine (or refinement) is referenced explicitly with a refinement clause (keyword: REFINES)
- A refinement may refine an abstract machine or a previous refinement
- The invariant establishes the type of the variables of the refinement and the refinement relation
- Turning an Abstract Machine into a more concrete one
  - preserving signature: name, parameters and results
  - a different state or a different specification of the operation

Example of an Abstract Machine

MACHINE

ExampleM

VARIABLES

y

INVARIANT

$y \in F(N_1)$

INITIALISATION

$y := \phi$

Example of an Abstract Machine (cont'd)

OPERATIONS

$enter(n) =$

PRE $n \in N_1$

THEN

$y := y \cup \{n\}$

END

$m \leftarrow getmax =$

PRE

$y \neq \phi$

THEN

$m := max(y)$

END;

END

Example of Refinement

REFINEMENT

ExampleR

REFINES

ExampleM

VARIABLES

z

INVARIANT

$z \in N \land z = max(y \cup \{0\})$

INITIALISATION

$z := 0$

Example of Refinement (cont'd)

OPERATIONS

$enter(n)$ =

PRE $n \in N_1$

THEN

$z := max(\{z, n\})$

END

$m \leftarrow getmax$ =

PRE

$z \neq 0$

THEN

$m := z$

END;

END

Verification of the Refinement

- The refinement has to be shown compliant with respect to the specification
  - the initialization of the refinement $INIT_R$ shall be compatible with the initialization of the specification $INIT_M$
  - the refined operations OP_R shall also be compatible with the specified operations OP_M
- compatible means that:
  - the concrete operations shall be possible whenever the corresponding specification is possible
  - the values established by the concrete initialization and operations shall be mapped, by the refinement relation INV_R, to a subset of those established in the specification

Substitution Refinement Pattern Formula

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- S being a substitution, R a predicate:
  - $[S]R$ means that all executions of S establish R
  - $\neg[S]\neg R$ means that there exists an execution of S establishing R
- Let INV_R be the refinement relation, S_M be a substitution on the abstract state, and S_R be a substitution on the concrete state, the formula:

  $$[S_R]\neg[S_M]\neg INV_R$$

  means that all executions of the concrete substitution S_R establish that there exists an execution of the abstract substitution S_M establishing INV_R

Proof Obligations for the Refinement's Initialization

- The invariant of the refinement INV_R defines the refinement relation
- It is assumed that the specification machine is consistent: the abstract invariant INV_M is established by all abstract initial states: $[INIT_M]INV_M$
- $[INIT_M]\neg INV_R$ characterizes the set of concrete states that are not related with an abstract initial state by INV_R
- The concrete initialization ($INIT_R$) shall guarantee that no concrete initial state satisfies $[INIT_M]\neg INV_R$, i.e. that the following formula is valid:

$$[INIT_R]\neg[INIT_M]\neg INV_R$$

Example of the Refinement's Initialization

$[INIT_R]\neg[INIT_M]\neg INV_R$

$$\begin{aligned}
&\Longleftrightarrow [z := 0]\neg[y := \phi]\neg(z \in N \wedge z = max(y \cup \{0\}))\\
&\Longleftrightarrow [z := 0]\neg\neg(z \in N \wedge z = max(\phi \cup \{0\}))\\
&\Longleftrightarrow [z := 0]z \in N \wedge z = max(\phi \cup \{0\})\\
&\Longleftrightarrow 0 \in N \wedge 0 = max(\{0\}) \;\; (using\;\; max(\{n\}) = n)\\
&\Longleftrightarrow 0 \in N \wedge 0 = 0\\
&\Longleftrightarrow \top
\end{aligned}$$

Proof Obligations for the Refinement's Output-less Operations

- The verification of the correctness when refining an operation without output is:

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow PRE_R \wedge [OP_R]\neg[OP_M]\neg INV_R$$

- PRE_M and OP_M are respectively the precondition and the substitution of the specified operation
- PRE_R and OP_R are respectively the precondition and the substitution of the refined operation

Example of the Refinement's Output-less Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow PRE_R \wedge [OP_R]\neg[OP_M]\neg INV_R$$

Example of the Refinement's Output-less Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow PRE_R \wedge [OP_R]\neg[OP_M]\neg INV_R$$

$$\begin{aligned}\iff & y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow \\ & n \in N_1 \wedge [z := max(\{z, n\})]\neg[y := y \cup \{n\}] \\ & \neg(z \in N \wedge z = max(y \cup \{0\}))\end{aligned}$$

Example of the Refinement's Output-less Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow PRE_R \wedge [OP_R]\neg[OP_M]\neg INV_R$$

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Example of the Refinement's Output-less Operations (cont'd)

$$\begin{aligned}\iff & y \in F(N_1) \land z \in N \land z = max(y \cup \{0\}) \land n \in N_1 \Rightarrow \\ & n \in N_1 \land [z := max(\{z, n\})] \\ & (z \in N \land z = max(y \cup \{n\} \cup \{0\}))\end{aligned}$$

Example of the Refinement's Output-less Operations (cont'd)

$$\iff y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow$$
$$n \in N_1 \wedge [z := max(\{z, n\})]$$
$$(z \in N \wedge z = max(y \cup \{n\} \cup \{0\}))$$

$$\iff y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow$$
$$n \in N_1 \wedge max(\{z, n\}) \in N \wedge$$
$$max(\{z, n\}) = max(y \cup \{n\} \cup \{0\}))$$

Example of the Refinement's Output-less Operations (cont'd)

$$\begin{aligned}
\iff & y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow \\
& n \in N_1 \wedge [z := max(\{z, n\})] \\
& (z \in N \wedge z = max(y \cup \{n\} \cup \{0\}))
\end{aligned}$$

$$\begin{aligned}
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& max(\{z, n\}) = max(y \cup \{n\} \cup \{0\}))
\end{aligned}$$

Example of the Refinement's Output-less Operations (cont'd)

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow max(\{z, n\}) = max(y \cup \{n\} \cup \{0\}))$$

(using $max(max(S1 \cup S2) \cup S3) = max(S1 \cup S2 \cup S3)$)

Example of the Refinement's Output-less Operations (cont'd)

$$\iff y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge n \in N_1 \Rightarrow$$
$$max(\{z, n\}) = max(y \cup \{n\} \cup \{0\}))$$

(using $max(max(S1 \cup S2) \cup S3) = max(S1 \cup S2 \cup S3)$)

$$\iff \top$$

Proof Obligations for the Refinement's Output Operations

- The verification of the correctness when refining an operation with output is:

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow$$
$$PRE_R \wedge [[o := o']OP_R]\neg[OP_M]\neg(INV_R \wedge o = o')$$

- o is the identifier of the output and o' is a fresh identifier.

Example of the Refinement's Output Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow$$
$$PRE_R \wedge [[o := o']OP_R]\neg[OP_M]\neg(INV_R \wedge o = o')$$

Example of the Refinement's Output Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow$$
$$PRE_R \wedge [[o := o']OP_R]\neg[OP_M]\neg(INV_R \wedge o = o')$$

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
$$z \neq 0 \wedge [[m := m']m := z]\neg([m := max(y)])$$
$$\neg(z \in N \wedge z = max(y \cup \{0\}) \wedge m = m')$$

Example of the Refinement's Output Operations

$$INV_M \wedge INV_R \wedge PRE_M \Rightarrow$$
$$PRE_R \wedge [[o := o']OP_R]\neg[OP_M]\neg(INV_R \wedge o = o')$$

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$$z \neq 0 \wedge [[m := m']m := z]$$
$$\neg\neg(z \in N \wedge z = max(y \cup \{0\}) \wedge max(y) = m')$$

Example of the Refinement's Output Operations (cont'd)

$$\iff y \in F(N_1) \land z \in N \land z = max(y \cup \{0\}) \land y \neq \phi \Rightarrow$$
$$z \neq 0 \land [[m := m']m := z]$$
$$(z \in N \land z = max(y \cup \{0\}) \land max(y) = m')$$

Example of the Refinement's Output Operations (cont'd)

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
$$z \neq 0 \wedge [[m := m']m := z]$$
$$(z \in N \wedge z = max(y \cup \{0\}) \wedge max(y) = m')$$

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
$$z \neq 0 \wedge [m' := z]$$
$$(z \in N \wedge z = max(y \cup \{0\}) \wedge max(y) = m')$$

Example of the Refinement's Output Operations (cont'd)

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
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Example of the Refinement's Output Operations (cont'd)

$$\iff y \in F(N_1) \land z \in N \land z = max(y \cup \{0\}) \land y \neq \phi \Rightarrow$$
$$z \neq 0 \land z \in N \land z = max(y \cup \{0\}) \land max(y) = z$$

Example of the Refinement's Output Operations (cont'd)

$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
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$$\Longleftrightarrow y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
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Example of the Refinement's Output Operations (cont'd)

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$$\iff y \in F(N_1) \wedge z \in N \wedge z = max(y \cup \{0\}) \wedge y \neq \phi \Rightarrow$$
$$z \neq 0 \wedge max(y) = z$$

$$\iff \top$$

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Generalities on Implementations in B

- Implementation is the input to programming language source code synthesis
- Implementation is a special case of refinement (same proof obligations)
- Implementation cannot be further refined
- Additional constraints:
  - Implementation cannot contain non-deterministic constructs
  - Implementation do not contain variables, but imports them from pre-existing B modules: library machines

Generalized Substitutions for Implementation

- The following substitutions provide support for programming in the B notation:
  - Sequencing: $S_1; S_2$
  - Choice: IF P THEN $S1$ ELSE $S2$ END
  - Loop: WHILE C DO S INVARIANT I VARIANT V END

Sequencing

- Syntax: $S_1; S_2$ denotes substitution S_1, followed by substitution S_2
- Semantics:
  $[S_1; S_2]\mathrm{P} \iff [S_1][S_2]\mathrm{P}$
- Sequencing is left associative:
  $S_1; S_2; S_3 = (S_1; S_2); S_3$

Choice

- Syntax: **IF** T **THEN** S_1 **ELSE** S_2
- Semantics:

  $$[\textbf{IF}\ \mathrm{T}\ \textbf{THEN}\ S_1\ \textbf{ELSE}\ S_2]\mathrm{P} \Longleftrightarrow (\ \mathbf{T} \Longrightarrow [S_1]\mathrm{P}) \wedge (\neg\ \mathbf{T} \Longrightarrow [S_2]\mathrm{P})$$

Construction of Loops

- Programming languages usually have several loop constructs
- The B notation provides the WHILE construct:
  - the control of the iteration is a test condition
  - the body of the loop is executed while the test is true
  - there is usually some initialization before the iteration

Loop Syntax

WHILE T : formula **DO** B : substitution
VARIANT V : expression **INVARIANT** I : formula
END

- The loop variant (rank function) states the maximum number of times that the body will be executed
- The loop invariant is a formula that shall be valid each time the control condition is evaluated

Loop Example

$y := x; ctr := 0;$
WHILE $ctr < 5$ **DO**
  $x := x + 1; ctr := ctr + 1$
VARIANT $6 - ctr$
INVARIANT $ctr \in 0..5 \wedge x = y + ctr$
END

Loop Verification

Loop verification:

$[INIT;\ \textbf{WHILE}\ T\ \textbf{DO}\ B\ \textbf{VARIANT}\ V\ \textbf{INVARIANT}\ I\ \textbf{END}]R$

Rules for the correctness of a loop:

- I-rule ($[INIT]I$)
- F-rule ($I \wedge \neg T \Rightarrow R$)
- T1-rule ($I \Rightarrow V \in N$)
- T2-rule ($I \wedge T \Rightarrow [V_i := V][B]V < V_i,\ V_i$ *is the initial variant*)
- P-rule ($I \wedge T \Rightarrow [B]I$)

Verifying a Loop

$[y := x; ctr := 0;$
$\text{WHILE } ctr < 5 \text{ DO}$
$x := x + 1; ctr := ctr + 1$
$\text{VARIANT } 6 - ctr$
$\text{INVARIANT } ctr \in 0..5 \wedge x = y + ctr$
$\text{END}]x = y + 5$

Verifying the I-rule

$$[INIT]I \iff [x := y; ctr := 0]ctr \in 0..5 \wedge x = y + ctr$$

Verifying the I-rule

$$
\begin{aligned}
[INIT]I &\iff [x := y; ctr := 0]ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \wedge x = y + ctr
\end{aligned}
$$

Verifying the I-rule

$$
\begin{aligned}
[INIT]I &\iff [x := y; ctr := 0]ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \wedge x = y + ctr
\end{aligned}
$$

Verifying the I-rule

$$
\begin{aligned}
[INIT]I &\iff [x := y; ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y]0 \in 0..5 \land x = y + 0
\end{aligned}
$$

Verifying the I-rule

$$
\begin{aligned}
[INIT]I &\iff [x := y; ctr := 0] ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y][ctr := 0] ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y][ctr := 0] ctr \in 0..5 \wedge x = y + ctr \\
&\iff [x := y] 0 \in 0..5 \wedge x = y + 0 \\
&\iff 0 \in 0..5 \wedge y = y + 0
\end{aligned}
$$

Verifying the I-rule

$$
\begin{aligned}
[INIT]I &\iff [x := y; ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y][ctr := 0]ctr \in 0..5 \land x = y + ctr \\
&\iff [x := y]0 \in 0..5 \land x = y + 0 \\
&\iff 0 \in 0..5 \land y = y + 0 \\
&\iff \top
\end{aligned}
$$

Verifying the F-rule

$$I \wedge \neg T \Rightarrow R \iff (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \Rightarrow x = y + 5$$

Verifying the F-rule

$$
\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5
\end{aligned}
$$

Verifying the F-rule

$$
\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5
\end{aligned}
$$

Verifying the F-rule

$$
\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5
\end{aligned}
$$

Verifying the F-rule

$$
\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
& \Rightarrow x = y + 5 \\
\iff & (ctr \geq 5 \wedge ctr \in 0..5 \wedge x = y + ctr) \\
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\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5 \\
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$$

Verifying the F-rule

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\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
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\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5 \\
\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5 \\
\iff & (x = y + 5) \Rightarrow x = y + 5
\end{aligned}
$$

Verifying the F-rule

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\begin{aligned}
I \wedge \neg T \Rightarrow R \iff & (\neg(ctr < 5) \wedge ctr \in 0..5 \wedge x = y + ctr) \\
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& \Rightarrow x = y + 5 \\
\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5 \\
\iff & (ctr = 5 \wedge x = y + ctr) \Rightarrow x = y + 5 \\
\iff & (x = y + 5) \Rightarrow x = y + 5 \\
\iff & \top
\end{aligned}
$$

Verifying the T1-rule

$$I \Rightarrow (V \in N) \Longleftrightarrow ctr \in 0..5 \wedge x = y + ctr \Rightarrow 6 - ctr \in N$$

Verifying the T1-rule

$$
\begin{aligned}
I \Rightarrow (V \in N) &\Longleftrightarrow ctr \in 0..5 \land x = y + ctr \Rightarrow 6 - ctr \in N \\
&\Longleftrightarrow \top
\end{aligned}
$$

Verifying the T2-rule

$$
\begin{aligned}
& I \wedge T \Rightarrow [V_i := V][B]V < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr][x := x + 1; ctr := ctr + 1]6 - ctr < V_i
\end{aligned}
$$

Verifying the T2-rule

$$I \wedge T \Rightarrow [V_i := V][B]V < V_i$$
$$\iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1; ctr := ctr + 1]6 - ctr < V_i$$
$$\iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i$$

Verifying the T2-rule

$$
\begin{aligned}
& I \wedge T \Rightarrow [V_i := V][B]V < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \quad [V_i := 6 - ctr][x := x + 1; ctr := ctr + 1]6 - ctr < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \quad [V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \quad [V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i
\end{aligned}
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Verifying the T2-rule

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\begin{aligned}
& I \wedge T \Rightarrow [V_i := V][B]V < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
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& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr][x := x + 1]6 - (ctr + 1) < V_i
\end{aligned}
$$

Verifying the T2-rule

$$I \wedge T \Rightarrow [V_i := V][B]V < V_i$$
$$\Longleftrightarrow (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1; ctr := ctr + 1]6 - ctr < V_i$$
$$\Longleftrightarrow (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i$$
$$\Longleftrightarrow (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1][ctr := ctr + 1]6 - ctr < V_i$$
$$\Longleftrightarrow (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr][x := x + 1]6 - (ctr + 1) < V_i$$
$$\Longleftrightarrow (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow$$
$$[V_i := 6 - ctr]6 - (ctr + 1) < V_i$$

Verifying the T2-rule

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\begin{aligned}
& I \wedge T \Rightarrow [V_i := V][B]V < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
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& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr][x := x + 1]6 - (ctr + 1) < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr]6 - (ctr + 1) < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad 5 - ctr < 6 - ctr
\end{aligned}
$$

Verifying the T2-rule

$$
\begin{aligned}
& I \wedge T \Rightarrow [V_i := V][B]V < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr][x := x + 1; ctr := ctr + 1]6 - ctr < V_i \\
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& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad [V_i := 6 - ctr]6 - (ctr + 1) < V_i \\
& \iff (ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5) \Rightarrow \\
& \qquad 5 - ctr < 6 - ctr \\
& \iff \top
\end{aligned}
$$

Verifying the P-rule

$$I \wedge T \Rightarrow [B]I \iff ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow$$
$$[x := x + 1; ctr := ctr + 1]ctr \in 0..5 \wedge x = y + ctr$$

Verifying the P-rule

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\begin{aligned}
I \wedge T \Rightarrow [B]I \iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1; ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr
\end{aligned}
$$

Verifying the P-rule

$$
\begin{aligned}
I \land T \Rightarrow [B]I \iff & ctr \in 0..5 \land x = y + ctr \land ctr < 5 \Rightarrow \\
& [x := x + 1; ctr := ctr + 1] ctr \in 0..5 \land x = y + ctr \\
\iff & ctr \in 0..5 \land x = y + ctr \land ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \land x = y + ctr \\
\iff & ctr \in 0..5 \land x = y + ctr \land ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \land x = y + ctr
\end{aligned}
$$

Verifying the P-rule

$$
\begin{aligned}
I \wedge T \Rightarrow [B]I \iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
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& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1] ctr + 1 \in 0..5 \wedge x = y + ctr + 1
\end{aligned}
$$

Verifying the P-rule

$$
\begin{aligned}
I \wedge T \Rightarrow [B]I \iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
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\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1] ctr + 1 \in 0..5 \wedge x = y + ctr + 1 \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1
\end{aligned}
$$

Verifying the P-rule

$$
\begin{aligned}
I \wedge T \Rightarrow [B]I \iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1; ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1] ctr + 1 \in 0..5 \wedge x = y + ctr + 1 \\
\iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1 \\
\iff & \; ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1
\end{aligned}
$$

Verifying the P-rule

IM NTU

$$
\begin{aligned}
I \wedge T \Rightarrow [B]I \iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1; ctr := ctr + 1]ctr \in 0..5 \wedge x = y + ctr \\
\iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1]ctr \in 0..5 \wedge x = y + ctr \\
\iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1]ctr \in 0..5 \wedge x = y + ctr \\
\iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1]ctr + 1 \in 0..5 \wedge x = y + ctr + 1 \\
\iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1 \\
\iff\ & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1 \\
\iff\ & x = y + ctr \Rightarrow x + 1 = y + ctr + 1
\end{aligned}
$$

Verifying the P-rule

$$
\begin{aligned}
I \wedge T \Rightarrow [B]I \iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1; ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1][ctr := ctr + 1] ctr \in 0..5 \wedge x = y + ctr \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& [x := x + 1] ctr + 1 \in 0..5 \wedge x = y + ctr + 1 \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1 \\
\iff & ctr \in 0..5 \wedge x = y + ctr \wedge ctr < 5 \Rightarrow \\
& ctr + 1 \in 0..5 \wedge x + 1 = y + ctr + 1 \\
\iff & x = y + ctr \Rightarrow x + 1 = y + ctr + 1 \\
\iff & \top
\end{aligned}
$$

Thanks for your attention!