Propositional Logic

(Based on [Gallier 1986], [Goubault-Larrecq and Mackie 1997], and [Huth and Ryan 2004])

Yih-Kuen Tsay

Department of Information Management
National Taiwan University

Introduction

- Logic concerns two concepts:
  - truth (in a specific or general context)
  - provability (of truth from assumed truth)
- *Formal (symbolic) logic* approaches logic by rules for manipulating symbols:
  - syntax rules: for writing statements or formulae.
    (There are also semantic rules determining whether a statement is true or false in a context or mathematical structure.)
  - inference rules: for obtaining true statements from other true statements.
- We shall introduce two main branches of formal logic:
  - *propositional logic*
  - *first-order logic* (predicate logic/calculus)
- The following slides cover propositional logic.

Propositions

- A *proposition* is a statement that is either *true* or *false* such as the following:
  - Leslie is a teacher.
  - Leslie is rich.
  - Leslie is a pop singer.
- Simplest (atomic) propositions may be combined to form compound propositions:
  - Leslie is *not* a teacher.
  - *Either* Leslie is not a teacher *or* Leslie is not rich.
  - *If* Leslie is a pop singer, *then* Leslie is rich.

Inferences

- We are given the following assumptions:
  - Leslie is a teacher.
  - Either Leslie is not a teacher or Leslie is not rich.
  - If Leslie is a pop singer, then Leslie is rich.
- We wish to conclude the following:
  - Leslie is not a pop singer.
- The above process is an example of *inference* (deduction). Is it correct?

Symbolic Propositions

- Propositions are represented by *symbols*, when only their truth values are of concern.
  - P: Leslie is a teacher.
  - Q: Leslie is rich.
  - R: Leslie is a pop singer.
- Compound propositions can then be more succinctly written.
  - *not* P: Leslie is not a teacher.
  - *not* P *or not* Q: Either Leslie is not a teacher or Leslie is not rich.
  - R *implies* Q: If Leslie is a pop singer, then Leslie is rich.

Symbolic Inferences

- We are given the following assumptions:
  - *P* (Leslie is a teacher.)
  - *not P or not Q* (Either Leslie is not a teacher or Leslie is not rich.)
  - *R implies Q* (If Leslie is a pop singer, then Leslie is rich.)
- We wish to conclude the following:
  - *not R* (Leslie is not a pop singer.)
- Correctness of the inference may be checked by asking:
  - Is (*P and* (*not P or not Q*) *and* (*R implies Q*)) *implies* (*not R*) a tautology (valid formula)?
  - Or, is (*A and* (*not A or not B*) *and* (*C implies B*)) *implies* (*not C*) a tautology (valid formula)?

Propositional Logic: Syntax

- Vocabulary:
  - A countable set $\mathcal{P}$ of *proposition symbols* (variables): $P, Q, R, \ldots$ (also called *atomic propositions*);
  - *Logical connectives* (operators): $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ and sometimes the constant $\perp$ (*false*);
  - Auxiliary symbols: "(", ")".
- *Propositional Formulae*:
  - Any $A \in \mathcal{P}$ is a formula (and so is $\perp$).
  - If A and B are formulae, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.

Propositional Logic: Semantics

- The meanings of positional formulae may be conveniently summarized by the truth table:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

  The meaning of $\perp$ is always F (false).
- There is an implicit inductive definition in the table. We shall try to make this precise.

Truth Assignment and Valuation

- The semantics of propositional logic assigns a truth function to each propositional formula.
- Let *BOOL* be the set of truth values $\{T, F\}$.
- A *truth assignment* (valuation) is a function from $\mathcal{P}$ (the set of proposition symbols) to *BOOL*.
- Let *PROPS* be the set of all propositional formulae.
- A truth assignment v may be extended to a *valuation* function $\hat{v}$ from *PROPS* to *BOOL* as follows:

Truth Assignment and Valuation (cont.)

$$
\begin{aligned}
\hat{v}(\bot) &= F \\
\hat{v}(P) &= v(P) \text{ for all } P \in \mathcal{P} \\
\hat{v}(P) &= \text{as defined by the table below, otherwise}
\end{aligned}
$$

$\hat{v}(A)$	$\hat{v}(B)$	$\hat{v}(\neg A)$	$\hat{v}(A \wedge B)$	$\hat{v}(A \vee B)$	$\hat{v}(A \rightarrow B)$	$\hat{v}(A \leftrightarrow B)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Truth Assignment and Satisfaction

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- We say $v \models A$ (v *satisfies* A) if $\hat{v}(A) = T$ and $v \not\models A$ (v *falsifies* A) if $\hat{v}(A) = F$.
- Alternatively, $\models$ may be defined as follows:

$$
\begin{array}{lcl}
v \not\models \bot & & \\
v \models P & \Longleftrightarrow & v(P) = T, \quad \text{for all } P \in \mathcal{P} \\
v \models \neg A & \Longleftrightarrow & v \not\models A \text{ (it is \textit{not} the case that } v \models A) \\
v \models A \wedge B & \Longleftrightarrow & v \models A \textit{ and } v \models B \\
v \models A \vee B & \Longleftrightarrow & v \models A \textit{ or } v \models B \\
v \models A \rightarrow B & \Longleftrightarrow & v \not\models A \textit{ or } v \models B \\
v \models A \leftrightarrow B & \Longleftrightarrow & (v \models A \textit{ and } v \models B) \\
 & & \textit{or } (v \not\models A \textit{ and } v \not\models B)
\end{array}
$$

Object vs. Meta Language

- The language that we study is referred to as the *object* language.
- The language that we use to study the object language is referred to as the *meta* language.
- For example, *not*, *and*, and *or* that we used to define the satisfaction relation $\models$ are part of the meta language.

Satisfiability

- A proposition A is *satisfiable* if there exists an assignment v such that $v \models A$.
  - $v(P) = F, v(Q) = T \models (P \vee Q) \wedge (\neg P \vee \neg Q)$
- A proposition is *unsatisfiable* if no assignment satisfies it.
  - $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge P$ is unsatisfiable.
- The problem of determining whether a given proposition is satisfiable is called the *satisfiability problem*.

Tautology and Validity

- A proposition A is a *tautology* if every assignment satisfies A, written as $\models A$.
  - $\models A \vee \neg A$
  - $\models (A \wedge B) \rightarrow (A \vee B)$
- The problem of determining whether a given proposition is a tautology is called the *tautology problem*.
- A proposition is also said to be *valid* if it is a tautology.
- So, the problem of determining whether a given proposition is valid (a tautology) is also called the *validity problem*.

Note: The notion of a tautology is restricted to propositional logic. In first-order logic, we also speak of valid formulae.

Validity vs. Satisfiability

Theorem

A proposition A is valid (a tautology) if and only if $\neg A$ is unsatisfiable.

So, there are two ways of proving that a proposition A is a tautology:

- A is satisfied by every truth assignment (or A *cannot be falsified* by any truth assignment).
- $\neg A$ is unsatisfiable.

Semantic Entailment

- Consider two sets of propositions Γ and Δ.
- We say that $v \models \Gamma$ (v satisfies Γ) if $v \models B$ for every $B \in \Gamma$; analogously for Δ.
- We say that Δ is a *semantic consequence* of Γ if every assignment that satisfies Γ also satisfies Δ, written as $\Gamma \models \Delta$.
  - $A, A \rightarrow B \models A, B$
  - $A \rightarrow B, \neg B \models \neg A$
- We also say that Γ *semantically entails* Δ when $\Gamma \models \Delta$.

Relating the Logical Connectives

Lemma

$$\models (A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A))$$

$$\models (A \rightarrow B) \leftrightarrow (\neg A \vee B)$$

$$\models (A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$$

$$\models \perp \leftrightarrow (A \wedge \neg A)$$

Note: These equivalences imply that some connectives could be dispensed with. We normally want a smaller set of connectives when analyzing properties of the logic and a larger set when actually using the logic.

Normal Forms

- A *literal* is an atomic proposition or its negation.
- A propositional formula is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions of literals.
  - $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q) \wedge P$
  - $(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
- A propositional formula is in Disjunctive Normal Form (DNF) if it is a disjunction of conjunctions of literals.
  - $(P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q) \vee P$
  - $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R)$
- A propositional formula is in Negation Normal Form (NNF) if negations occur only in literals.
  - CNF or DNF is also NNF (but not vice versa).
  - $(P \wedge \neg Q) \wedge (P \vee (Q \wedge \neg R))$ in NNF, but not CNF or DNF.

Sequents

- A (propositional) *sequent* is an expression of the form $\Gamma \vdash \Delta$, where $\Gamma = A_1, A_2, \cdots, A_m$ and $\Delta = B_1, B_2, \cdots, B_n$ are finite (possibly empty) sequences of (propositional) formulae.
- In a sequent $\Gamma \vdash \Delta$, Γ is called the *antecedent* (also *context*) and Δ the *consequent*.

Note: Many authors prefer to write a sequent as $\Gamma \longrightarrow \Delta$ or $\Gamma \Longrightarrow \Delta$, while reserving the symbol $\vdash$ for provability (deducibility) in the proof (deduction) system under consideration.

Sequents (cont.)

- A sequent $A_1, A_2, \cdots, A_m \vdash B_1, B_2, \cdots, B_n$ is falsifiable if there exists a valuation v such that $v \models (A_1 \wedge A_2 \wedge \cdots \wedge A_m) \wedge (\neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_n)$.
  - $A \vee B \vdash B$ is falsifiable, as $v(A) = T, v(B) = F \models (A \vee B) \wedge \neg B$.
- A sequent $A_1, A_2, \cdots, A_m \vdash B_1, B_2, \cdots, B_n$ is valid if, for every valuation v, $v \models A_1 \wedge A_2 \wedge \cdots \wedge A_m \rightarrow B_1 \vee B_2 \vee \cdots \vee B_n$.
  - $A \vdash A, B$ is valid.
  - $A, B \vdash A \wedge B$ is valid.
- A sequent is valid if and only if it is not falsifiable.
- In the following, we will use only sequents of this simpler form: $A_1, A_2, \cdots, A_m \vdash C$, where C is a formula.

Inference Rules

- Inference rules allow one to obtain true statements from other true statements.
- Below is an inference rule for conjunction.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \ (\wedge I)$$

- In an inference rule, the upper sequents (above the horizontal line) are called the *premises* and the lower sequent is called the *conclusion*.

Proofs

- A deduction tree is a tree where each node is labeled with a sequent such that, for every internal (non-leaf) node,
  - the label of the node corresponds to the conclusion and
  - the labels of its children correspond to the premises

  of an instance of an inference rule.
- A proof tree is a deduction tree, each of whose leaves is labeled with an axiom.
- The root of a deduction or proof tree is called the conclusion.
- A sequent is provable if there exists a proof tree of which it is the conclusion.

Natural Deduction in the Sequent Form

$$\frac{}{\Gamma, A \vdash A}\ (Ax)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge I)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee I_1)$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee I_2)$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee E)$$

Natural Deduction (cont.)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} (\to I) \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B} (\to E)$$

$$\frac{\Gamma, A \vdash B \wedge \neg B}{\Gamma \vdash \neg A} (\neg I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B} (\neg E)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} (\neg\neg I) \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} (\neg\neg E)$$

These inference rules collectively are called System *ND* (the propositional part).

Soundness and Completeness

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Theorem

System ND is sound, i.e., if a sequent $\Gamma \vdash C$ is provable in ND, then $\Gamma \vdash C$ is valid.

Theorem

System ND is complete, i.e., if a sequent $\Gamma \vdash C$ is valid, then $\Gamma \vdash C$ is provable in ND.

Compactness

A set Γ of propositions is satisfiable if some valuation satisfies every proposition in Γ. For example, $\{A \vee B, \neg B\}$ is satisfiable.

Theorem

For any (possibly infinite) set Γ of propositions, if every finite non-empty subset of Γ is satisfiable then Γ is satisfiable.

Proof hint: by contradiction and the completeness of *ND*.

Consistency

- A set Γ of propositions is *consistent* if there exists some proposition B such that the sequent $\Gamma \vdash B$ is not provable.
- Otherwise, Γ is *inconsistent*; e.g., $\{A, \neg(A \vee B)\}$ is inconsistent.

Lemma

For System ND, a set Γ of propositions is inconsistent if and only if there is some proposition A such that both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ are provable.

Theorem

For System ND, a set Γ of propositions is satisfiable if and only if Γ is consistent.