

# Verification of Sequential Programs: Hoare Logic (I)

(Based on [Apt and Olderog 1991; Gries 1981;  
Hoare 1969; Kleymann 1999; Sethi 1996])

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# An Axiomatic View of Programs

- 🌐 The **properties** of a program can, in principle, be found out from its text by means of purely *deductive reasoning*.
- 🌐 The deductive reasoning involves the application of valid *inference rules* to a set of valid *axioms*.
- 🌐 The choice of axioms will depend on the choice of programming languages.
- 🌐 We shall introduce such an axiomatic approach, called the *Hoare logic*, to program correctness.

- 🌐 When executed, a program will evolve through different *states*, which are essentially a mapping of the program variables to values in their respective domains.
- 🌐 To reason about correctness of a program, we inevitably need to talk about its states.
- 🌐 An *assertion* is a precise statement about the state of a program.
- 🌐 Most interesting assertions can be expressed in a *first-order* language.

# Pre and Post-conditions

- 🌐 The behavior of a “structured” (single-entry/single-exit) program statement can be characterized by **attaching assertions at the entry and the exit of the statement**.
- 🌐 For a statement  $S$ , this is conveniently expressed as a so-called *Hoare triple*, denoted  $\{P\} S \{Q\}$ , where
  - ☀  $P$  is called the *pre-condition* and
  - ☀  $Q$  is called the *post-condition* of  $S$ .

# Interpretations of a Hoare Triple

🌐 A Hoare triple  $\{P\} S \{Q\}$  may be interpreted in two different ways:

- ☀️ **Partial Correctness:** if the execution of  $S$  starts in a state satisfying  $P$  and terminates, then it results in a state satisfying  $Q$ .
- ☀️ **Total Correctness:** if the execution of  $S$  starts in a state satisfying  $P$ , then it will terminate and result in a state satisfying  $Q$ .

Note: sometimes we write  $\langle P \rangle S \langle Q \rangle$  when total correctness is intended.

# Pre and Post-Conditions for Specification

- Find an integer approximate to the square root of another integer  $n$ :

$$\{0 \leq n\} ? \{d^2 \leq n < (d + 1)^2\}$$

or slightly better (clearer about what can be changed)

$$\{0 \leq n\} \ d := ? \ \{d^2 \leq n < (d + 1)^2\}$$

- Find the index of value  $x$  in an array  $b$ :

- $\{x \in b[0..n - 1]\} ? \{0 \leq i < n \wedge x = b[i]\}$




- $\{0 \leq n\} ? \{(0 \leq i < n \wedge x = b[i]) \vee (i = n \wedge x \notin b[0..n - 1])\}$

Note: there are other ways to stipulate which variables are to be changed and which are not.

# A Little Bit of History

The following seminal paper started it all:

*C.A.R. Hoare. An axiomatic basis for computer programs. CACM, 12(8):576-580, 1969.*

-  Original notation:  $P \{S\} Q$  (vs.  $\{P\} S \{Q\}$ )
-  Interpretation: partial correctness
-  Provided axioms and proof rules

Note: R.W. Floyd did something similar for flowcharts earlier in 1967, which was also a precursor of “proof outline” (a program fully annotated with assertions).

# The Assignment Statement

🌐 Syntax:

$$x := E$$

- 🌐 Meaning: execution of the assignment  $x := E$  (read as “ $x$  becomes  $E$ ”) evaluates  $E$  and stores the result in variable  $x$ .
- 🌐 We will assume that expression  $E$  in  $x := E$  has *no side-effect* (i.e., does not change the value of any variable).
- 🌐 Which of the following two Hoare triples is correct about the assignment  $x := E$ ?




☀️  $\{P\} x := E \{P[E/x]\}$

☀️  $\{Q[E/x]\} x := E \{Q\}$

Note:  $E$  is essentially a first-order term.



# Some Hoare Triples for Assignments

-   $\{x > 0\} x := x - 1 \{x \geq 0\}$   
or equivalently,  $\{x - 1 \geq 0\} x := x - 1 \{x \geq 0\}$
-   $\{x + 1 > 5\} x := x + 1 \{x > 5\}$
-   $\{5 \neq 5\} x := 5 \{x \neq 5\}$

# Axiom of the Assignment Statement

$$\frac{}{\{Q[E/x]\} x := E \{Q\}} \text{ (Assignment)}$$

Why is this so?

- 🌐 Let  $s$  be the state **before**  $x := E$  and  $s'$  the state **after**.
- 🌐 So,  $s' = s[x := E]$  assuming  $E$  has no side-effect.
- 🌐  $Q[E/x]$  holds in  $s$  if and only if  $Q$  holds in  $s'$ , because
  - ☀ every variable, except  $x$ , in  $Q[E/x]$  and  $Q$  has the same value in  $s$  and  $s'$ , and
  - ☀  $Q[E/x]$  has every  $x$  in  $Q$  replaced by  $E$ , while  $Q$  has every  $x$  evaluated to  $E$  in  $s'$  ( $= s[x := E]$ ).

# The Multiple Assignment Statement

## 🌐 Syntax:

$$x_1, x_2, \dots, x_n := E_1, E_2, \dots, E_n$$

where  $x_i$ 's are distinct variables.

🌐 Meaning: execution of the multiple assignment evaluates all  $E_i$ 's and stores the results in the corresponding variables  $x_i$ 's.

## 🌐 Examples:

☀️  $i, j := 0, 0$  (initialize  $i$  and  $j$  to 0)

☀️  $x, y := y, x$  (swap  $x$  and  $y$ )

☀️  $g, p := g + 1, p - 1$  (increment  $g$  by 1, while decrement  $p$  by 1)

☀️  $i, x := i + 1, x + i$  (increment  $i$  by 1 and  $x$  by  $i$ )

# Some Hoare Triples for Multi-assignments

## 🌐 Swapping two values

$$\{x < y\} x, y := y, x \{y < x\}$$

## 🌐 Number of games in a tournament

$$\{g + p = n\} g, p := g + 1, p - 1 \{g + p = n\}$$







## 🌐 Taking a sum

$$\{x + i = 1 + 2 + \dots + (i + 1 - 1)\}$$

$$i, x := i + 1, x + i$$

$$\{x = 1 + 2 + \dots + (i - 1)\}$$

# Simultaneous Substitution

-   $P[E/x]$  can be naturally extended to allow  $E$  to be a list  $E_1, E_2, \dots, E_n$  and  $x$  to be  $x_1, x_2, \dots, x_n$ , all of which are distinct variables.
-   $P[E/x]$  is then the result of simultaneously replaying  $x_1, x_2, \dots, x_n$  with the corresponding expressions  $E_1, E_2, \dots, E_n$ ; enclose  $E_i$ 's in parentheses if necessary.
-  Examples:
  -   $(x < y)[y, x/x, y] = (y < x)$
  -   $(g + p = n)[g + 1, p - 1/g, p] = ((g + 1) + (p - 1) = n) = (g + p = n)$
  -   $(x = 1 + 2 + \dots + (i - 1))[i + 1, x + i/i, x]$   
 $= ((x + i) = 1 + 2 + \dots + ((i + 1) - 1))$   
 $= (x + i = 1 + 2 + \dots + ((i + 1) - 1))$

# Axiom of the Multiple Assignment

🌐 Syntax:

$$x_1, x_2, \dots, x_n := E_1, E_2, \dots, E_n$$

where  $x_i$ 's are distinct variables.

🌐 Axiom:

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$$\frac{\{Q[E_1, \dots, E_n/x_1, \dots, x_n]\}}{x_1, \dots, x_n := E_1, \dots, E_n \{Q\}} \quad (\text{Assign.})$$

# Assignment to an Array Entry

🌐 Syntax:

$$b[i] := E$$

🌐 Notation for an altered array:  $(b; i : E)$  denotes the array that is identical to  $b$ , except that entry  $i$  stores the value of  $E$ .

$$(b; i : E)[j] = \begin{cases} E & \text{if } i = j \\ b[j] & \text{if } i \neq j \end{cases}$$

🌐 Axiom:

$$\frac{}{\{Q[(b; i : E)/b]\} b[i] := E \{Q\}} \text{ (Assignment)}$$

# Pre and Post-condition of a Loop

- 🌐 A precondition just **before** a loop can capture the conditions for executing the loop.
- 🌐 An assertion just **within** a loop body can capture the conditions for staying in the loop.
- 🌐 A postcondition just **after** a loop can capture the conditions upon leaving the loop.



# A Simple Example

 $\{x \geq 0 \wedge y > 0\}$ **while**  $x \geq y$  **do** $\{x \geq 0 \wedge y > 0 \wedge x \geq y\}$  $x := x - y$ **od** $\{x \geq 0 \wedge y > 0 \wedge x \not\geq y\}$ 

// or

 $\{x \geq 0 \wedge y > 0 \wedge x < y\}$

## More about the Example

We can say more about the program.

```
// may assume  $x, y := m, n$  here for some  $m \geq 0$  and  $n > 0$   
{ $x \geq 0 \wedge y > 0 \wedge (x \equiv m \pmod{y})$ }  
while  $x \geq y$  do  
     $x := x - y$   
od  
{ $x \geq 0 \wedge y > 0 \wedge (x \equiv m \pmod{y}) \wedge x < y$ }
```

Note: repeated subtraction is a way to implement the integer division. So, the program is taking the residue of  $x$  divided by  $y$ .

# A Simple Programming Language

- To study inference rules of Hoare logic, we consider a simple programming language with the following syntax for statements:

$$S ::= \begin{array}{l} \mathbf{skip} \\ | x := E \\ | S_1; S_2 \\ | \mathbf{if } B \mathbf{ then } S \mathbf{ fi} \\ | \mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi} \\ | \mathbf{while } B \mathbf{ do } S \mathbf{ od} \end{array}$$

# Proof Rules

$$\frac{}{\{Q[E/x]\} x := E \{Q\}}$$

(Assignment)

$$\frac{}{\{P\} \text{skip} \{P\}}$$

(Skip)

$$\frac{\{P\} S_1 \{Q\} \quad \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$$

(Sequence)

$$\frac{\{P \wedge B\} S_1 \{Q\} \quad \{P \wedge \neg B\} S_2 \{Q\}}{\{P\} \text{if } B \text{ then } S_1 \text{ else } S_2 \text{ fi} \{Q\}}$$

(Conditional)

“if  $B$  then  $S$  fi” can be treated as “if  $B$  then  $S$  else skip fi” or directly with the following rule:

$$\frac{\{P \wedge B\} S \{Q\} \quad P \wedge \neg B \rightarrow Q}{\{P\} \text{if } B \text{ then } S \text{ fi} \{Q\}}$$

(If-Then)

# Proof Rules (cont.)

$$\frac{\{P \wedge B\} S \{P\}}{\{P\} \mathbf{while} B \mathbf{do} S \mathbf{od} \{P \wedge \neg B\}} \quad \text{(While)}$$

$$\frac{P \rightarrow P' \quad \{P'\} S \{Q'\} \quad Q' \rightarrow Q}{\{P\} S \{Q\}} \quad \text{(Consequence)}$$

Note: with a suitable notion of validity, the set of proof rules up to now can be shown to be **sound** and (relatively) **complete** for programs that use only the considered constructs.

## Some Auxiliary Rules

$$\frac{P \rightarrow P' \quad \{P'\} S \{Q\}}{\{P\} S \{Q\}}$$

(Strengthening Precondition)

$$\frac{\{P\} S \{Q'\} \quad Q' \rightarrow Q}{\{P\} S \{Q\}}$$

(Weakening Postcondition)

$$\frac{\{P_1\} S \{Q_1\} \quad \{P_2\} S \{Q_2\}}{\{P_1 \wedge P_2\} S \{Q_1 \wedge Q_2\}}$$

(Conjunction)

$$\frac{\{P_1\} S \{Q_1\} \quad \{P_2\} S \{Q_2\}}{\{P_1 \vee P_2\} S \{Q_1 \vee Q_2\}}$$

(Disjunction)

Note: these rules provide more convenience, but do not actually add deductive power.

# Invariants

- 🌐 An *invariant* at some point of a program is an assertion that holds whenever execution of the program reaches that point.
- 🌐 Assertion  $P$  in the rule for a while loop is called a *loop invariant* of the while loop.
- 🌐 An assertion is called an *invariant of an operation* (a segment of code) if, assumed true before execution of the operation, the assertion remains true after execution of the operation.
- 🌐 Invariants are a bridge between the **static text** of a program and its **dynamic computation**.

# Program Annotation

- Inserting assertions/invariants in a program as comments helps understanding of the program.

$$\{x \geq 0 \wedge y > 0 \wedge (x \equiv m \pmod{y})\}$$

**while**  $x \geq y$  **do**

$$\{x \geq 0 \wedge y > 0 \wedge x \geq y \wedge (x \equiv m \pmod{y})\}$$
$$x := x - y$$
$$\{y > 0 \wedge x \geq 0 \wedge (x \equiv m \pmod{y})\}$$

**od**

$$\{x \geq 0 \wedge y > 0 \wedge (x \equiv m \pmod{y}) \wedge x < y\}$$

- A correct annotation of a program can be seen as a partial **proof outline** for the program.
- Boolean assertions can also be used as an aid to program testing.



# An Annotated Program

```
{x ≥ 0 ∧ y ≥ 0 ∧ gcd(x, y) = gcd(m, n)}  
while x ≠ 0 and y ≠ 0 do  
  {x ≥ 0 ∧ y ≥ 0 ∧ gcd(x, y) = gcd(m, n)}  
  if x < y then x, y := y, x fi;  
  {x ≥ y ∧ y ≥ 0 ∧ gcd(x, y) = gcd(m, n)}  
  x := x - y  
  {x ≥ 0 ∧ y ≥ 0 ∧ gcd(x, y) = gcd(m, n)}  
od  
{(x = 0 ∧ y ≥ 0 ∧ y = gcd(x, y) = gcd(m, n)) ∨  
 (x ≥ 0 ∧ y = 0 ∧ x = gcd(x, y) = gcd(m, n))}
```

Note:  $m$  and  $n$  are two arbitrary non-negative integers, at least one of which is nonzero.

# Total Correctness: Termination

🌐 All inference rules introduced so far, except the **while** rule, work for total correctness.

🌐 Below is a rule for the total correctness of the **while** statement:




$$\frac{\{P \wedge B\} S \{P\} \quad \{P \wedge B \wedge t = Z\} S \{t < Z\} \quad P \rightarrow (t \geq 0)}{\{P\} \mathbf{while} B \mathbf{do} S \mathbf{od} \{P \wedge \neg B\}}$$

where  $t$  is an integer-valued expression (state function) and  $Z$  is a “rigid” variable that does not occur in  $P$ ,  $B$ ,  $t$ , or  $S$ .

🌐 The above function  $t$  is called a *rank* (or variant) function.

# Termination of a Simple Program

```
 $g, p := 0, n; \quad // \quad n \geq 1$   
while  $p \geq 2$  do  
     $g, p := g + 1, p - 1$   
od
```

-  Loop Invariant:  $(g + p = n) \wedge (p \geq 1)$
-  Rank (Variant) Function:  $p$
-  The loop terminates when  $p = 1$  ( $p \geq 1 \wedge p \not\geq 2$ ).

# Well-Founded Sets

- 🌐 A binary relation  $\preceq \subseteq A \times A$  is a **partial order** if it is
  - ☀ reflexive:  $\forall x \in A (x \preceq x)$ ,
  - ☀ transitive:  $\forall x, y, z \in A ((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z)$ , and
  - ☀ antisymmetric:  $\forall x, y \in A ((x \preceq y \wedge y \preceq x) \rightarrow x = y)$ .
- 🌐 A partially ordered set  $(W, \preceq)$  is **well-founded** if there is no infinite decreasing chain  $x_1 \succ x_2 \succ x_3 \succ \dots$  of elements from  $W$ . (Note: “ $x \succ y$ ” means “ $y \preceq x \wedge y \neq x$ ”.)
- 🌐 Examples:
  - ☀  $(\mathbb{Z}_{\geq 0}, \leq)$
  - ☀  $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, \leq)$ ,  
where  $(x_1, y_1) \leq (x_2, y_2)$  if  $(x_1 < x_2) \vee (x_1 = x_2 \wedge y_1 \leq y_2)$

# Termination by Well-Founded Induction

Below is a more general rule for the total correctness of the **while** statement:

$$\frac{\{P \wedge B\} S \{P\} \quad \{P \wedge B \wedge \delta = D\} S \{\delta \prec D\} \quad P \rightarrow (\delta \in W)}{\{P\} \mathbf{while} B \mathbf{do} S \mathbf{od} \{P \wedge \neg B\}}$$

where  $(W, \preceq)$  is a **well-founded** set,  $\delta$  is a state function, and  $D$  is a “rigid” variable ranged over  $W$  that does not occur in  $P$ ,  $B$ ,  $\delta$ , or  $S$ .

# Nondeterminism

 Syntax of the Alternative Statement:

**if**  $B_1 \rightarrow S_1$

||  $B_2 \rightarrow S_2$

...

||  $B_n \rightarrow S_n$

**fi**

Each of the " $B_i \rightarrow S_i$ "s is called a **guarded command**, where  $B_i$  is the guard of the command and  $S_i$  the body.

 Semantic:

1. One of the guarded commands, whose guard evaluates to true, is nondeterministically selected and its body executed.
2. If none of the guards evaluates to true, then the execution aborts.

# Rule for the Alternative Statement

🌐 The Alternative Statement:

```
if  $B_1 \rightarrow S_1$   
   $\parallel B_2 \rightarrow S_2$   
  ...  
   $\parallel B_n \rightarrow S_n$   
fi
```

🌐 Inference rule:

$$\frac{P \rightarrow B_1 \vee \dots \vee B_n \quad \{P \wedge B_i\} S_i \{Q\}, \text{ for } 1 \leq i \leq n}{\{P\} \text{ **if** } B_1 \rightarrow S_1 \parallel \dots \parallel B_n \rightarrow S_n \text{ **fi** } \{Q\}}$$

# The Coffee Can Problem as a Program

$B, W := m, n; \quad // \quad m > 0 \wedge n > 0$

**while**  $B + W \geq 2$  **do**




**if**  $B \geq 0 \wedge W > 1 \rightarrow B, W := B + 1, W - 2 \quad // \text{ same color}$

$\parallel B > 1 \wedge W \geq 0 \rightarrow B, W := B - 1, W \quad // \text{ same color}$

$\parallel B > 0 \wedge W > 0 \rightarrow B, W := B - 1, W \quad // \text{ different colors}$

**fi**

**od**

-  Loop Invariant:  $W \equiv n \pmod{2}$  (and  $B + W \geq 1$ )
-  Variant (Rank) Function:  $B + W$
-  The loop terminates when  $B + W = 1$ .



# References

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